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On Λ^{λ} -Homeomorphisms In Topological Spaces

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Abstract

In this paper, we first introduce a new class of closed map called Λ^{λ} -closed map. Moreover, we introduce a new class of homeomorphism called Λ^{λ} - Homeomorphism, which are weaker than homeomorphism. We also introduce Λ^{λ^*} - Homeomorphisms and prove that the set of all Λ^{λ} - Homeomorphisms form a group under the operation of composition of maps.

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1. Introduction

The notion homeomorphism plays a very important role in topology. By definition, a homeomorphism between two topological spaces X and Y is a bijective map f: X $\to$ Y when both and f and f^{-1} are continuous. Maki. et al. [5] introduced
g-homeomorphisms and gc-homeomorphisms in topological spaces.

In this paper, we first introduce Λ^{λ} - closed maps in topological spaces and then we introduce and study Λ^{λ}-homeomorphisms, which are weaker than homeomorphisms. We also introduce Λ^{λ^*}- homeomorphisms. It turns out that the set of all Λ^{λ^*}- homeomorphisms forms a group under the operation composition of functions.

2. Preliminaries

Throughout this paper (X ,τ) and (Y,σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), Cl (A), Int (A) ant X - A denote the closure of A, the interior of A and the complement of A in X, respectively.

We recall the following definitions and some results, which are used in the sequel.

Definition 2.1: A subset A of a space (X, τ) is called:-

(1) λ- closed [1] if A = B$\cap$C, where B is a Λ- set and C is a closed set.

The complement of λ– closed set is called a λ – open set.

(2) Λ-g-closed [2] if Cl$_{\lambda}$ (A)$\subseteq U$, wherever A$\subseteq U$, U is λ-open, where Cl$_{\lambda}$ (A) [4] is called the λ - Closure of A. The complement of Λ-g-closed set is called a Λ-g-closed – open set.

The family of all λ - open subsets of a space (X, τ) shall be denoted by
λO (X, τ).

Definition 2.2:

A subset A of a space (X, τ) is called a generalized closed (briefly g-closed) set [6] if Cl (A) $\subset$ U whenever A $\subset$ U and U is open in (X, τ) .

Definition 2.5: A function f : (X, τ) →(Y,σ) is called :

(1) λ-continuous [1] if f^{-1} (V) is λ-open in (X,τ), for every open set V in (Y,σ).

(2) λ-irresolute [3] if f^{-1} (V) is λ-open in (X,τ), for every λ open set V in (Y,σ).

(3) g-continuous [6] if f^{-1} (V) is g-closed in (X,τ) for every g-closed set V in (Y,σ).

(4) gc-irresolute [6] if f^{-1} (V) is g-closed in (X,τ) for every g-closed set V in (Y,σ).

(5) λ-closed [3] if f (V) is λ-closed in (Y,σ), for every closed set V in (X,τ).

Definition 2.6: A function f : (X,τ) →(Y,σ) is called g-open [6] if f (V) is g-open in (Y,σ), for every g-open set V in (X,τ).

Definition 2.7: A bijective function f : (X,τ) →(Y,σ) is called a:

(1) generalized homeomorphism (briefly g-homeomorphism) [7] if f is both g-continuous and g-open.

(2) gc-homeomorphism [7] if both f and f^{-1} are gc-irresolute maps.

3. Λ^{λ}-Closed Sets

Definition 3.1: A subset A of a topological space (X, τ) is called :

(1) Λ_*^{λ}- set if A = $A^{\Lambda_*^{\lambda}}$, where $A^{\Lambda_*^{\lambda}}$ = $\cap$ {B : B$\supset$A, B$\in \lambda o$(X, τ)}

(2) Λ^{λ}- closed set if A= L$\cap$F, where L is Λ_*^{λ} - set and F is λ- closed.

The complement of Λ_*^{λ} -set and Λ^{λ}-closed set is V_*^{λ}-set and Λ^{λ}-open set respectively.

Proposition 3.2:

1) [3]Every closed (resp.open) set is λ-closed (resp. λ-open) set.

2) Every λ-closed (resp. λ-open) set is Λ^{λ}-closed (resp. Λ^{λ}-open) set.

Definition 3.3:A function f : (X, τ) →(Y,σ) is called :

1) Λ-g-continuous if f^{-1} (V) is Λ-g-open in (X,τ), for every open set V in (Y,σ).

2) Λ^{λ}-continuous if f^{-1} (V) is Λ^{λ}-open in (X,τ), for every open set V in (Y,σ).

3) $\Lambda^{\lambda}-\lambda$-continuous if f^{-1} (V) is Λ^{λ}-open in (X,τ), for every λ-open set V in (Y,σ).

4) Λ^{λ}-irresolute if f^{-1} (V) is Λ^{λ}-open in (X,τ), for every Λ^{λ}-open set V in (Y,σ).

5) $\Lambda^{\lambda*}$-open if f (V) is Λ^{λ}-open in (Y,σ), for every Λ^{λ}-open set V in (X, τ).

Definition 3.4: Let (X,τ) be a space and A $\subset$X. A Point x$\in$ X is called Λ^{λ}-cluster point of A if for every Λ^{λ}-open set U of X containing x, A$\cap$U$\neq \phi$. The set of all Λ^{λ}-cluster points is called the Λ^{λ}- closure of A and is denoted by $Cl^{\Lambda^{\lambda}}$(A).

Theorem 3.5: If f : (X,τ)→(Y,σ) is Λ^{λ} - irresolute then the map f is Λ^{λ} - continuous.

Proof: Let *f* be Λ^{λ} - irresolute Let V be an open set in (Y,σ). By Proposition 3.2, V is Λ^{λ} -open in (Y,σ). Since *f* is Λ^{λ} - irresolute, f^{-1}(V) is Λ^{λ} -open in (X,τ). Hence *f* is Λ^{λ} - continuous.

Proposition 3.6: Let A and B be a subset of a topological space (X,τ). The following properties hold:

(1) $A \subset Cl^{\Lambda^{\lambda}}(A) \subset Cl^{\lambda}(A)$

(2) $Cl^{\Lambda^{\lambda}}$(A)= $\cap$ {F $\in \Lambda^{\lambda}$C (X, τ)/ A$\subset$F}

(3) If $A \subset B$ then $Cl^{\Lambda^{\lambda}}$(A) $\subset$ $Cl^{\Lambda^{\lambda}}$(B)

(4) A is Λ^{λ} - closed if and only if A= $Cl^{\Lambda^{\lambda}}$(A).

(5) $Cl^{\Lambda^{\lambda}}$(A) is Λ^{λ} -closed .

Proof:

(1) Let $x \notin Cl^{\Lambda^{\lambda}}(A)$. Then *x* is not a Λ^{λ} -cluster point of A. So there exists a Λ^{λ} -open set U containing *x* such that $A \cap U = \phi$ and hence $x \notin A$.

Then $Cl^{\Lambda^{\lambda}}(A) \subset Cl^{\lambda}(A)$ follows from Proposition 3.2.

(2) Suppose $x \in \cap\{F / A \subset F \text{ and } F \text{ is } \Lambda^{\lambda} - closed\}$.Let U be a Λ^{λ} -open set containing *x* such that $A \cap U = \phi$.Andso $A \subset X - U$. But X-U is Λ^{λ} -closed and hence $Cl^{\Lambda^{\lambda}}(A) \subset X - U$.Since $x \notin X - U$, we obtain $x \notin Cl^{\Lambda^{\lambda}}(A)$
which is contrary to the hypothesis. Hence
$Cl^{\Lambda^{\lambda}}(A) \supset \cap\{F / A \subset F \text{ and } F \text{ is } \Lambda^{\lambda} - closed\}$.

Suppose that $x \in Cl^{\Lambda^{\lambda}}(A)$, i.e., that every Λ^{λ} -open set of X containing *x* meets
A. If $x \notin \cap\{F / A \subset F \text{ and } F \text{ is } \Lambda^{\lambda} - closed\}$, then there exists a Λ^{λ} -closed set

F of X such that $A \subset F$ and $x \notin F$. Therefore $x \in X - F \in \Lambda^{\lambda}O(X,\tau)$. Hence X-F is a Λ^{λ} -open set of X containing *x*, but $(X - F) \cap A = \phi$. But this is a contradiction. Hence $Cl^{\Lambda^{\lambda}}(A) \subset \cap\{F / A \subset F \text{ and } F \text{ is } \Lambda^{\lambda} - closed\}$. Thus,

$Cl^{\Lambda^\lambda}(A) = \cap\{F / A \subset F \ and \ F \ is \ \Lambda^\lambda - closed\}.$

(3) Let $x \notin Cl^{\Lambda^\lambda}(B)$. Then there exists a Λ^λ-open set V containing *x* such that $B \cap V = \phi$. Since $A \subset B, A \cap V = \phi$ and hence *x* is not a Λ^λ-cluster point of A. Therefore $x \notin Cl^{\Lambda^\lambda}(A)$.

(4) Let A is Λ^λ closed. Let $x \notin A$ Then *x* belongs to the Λ^λ-open X-A. Then a Λ^λ-open set X-A containing *x* and $A \cap (X - A) = \phi$.Hence $x \notin Cl^{\Lambda^\lambda}(A)$. By (1),we get $A = Cl^{\Lambda^\lambda}(A)$. Conversely, Suppose $A = Cl^{\Lambda^\lambda}(A)$. By (2)

A=$\cap$ {F $\in \Lambda^\lambda$ C (X, τ)/ A$\subset$F}.Hence A is Λ^λ-closed .

(5) By (1) and (3), we have $Cl^{\Lambda^\lambda}(A) \subset Cl^{\Lambda^\lambda}(Cl^{\Lambda^\lambda}(A))$. Let $x \in Cl^{\Lambda^\lambda}(Cl^{\Lambda^\lambda}(A))$.Hence *x* is a Λ^λ-cluster point of $Cl^{\Lambda^\lambda}(A)$. That implies for every Λ^λ-open set U containing *x*,$Cl^{\Lambda^\lambda}(A) \cap U \neq \phi$.Let $p \in Cl^{\Lambda^\lambda}(A) \cap U$. Then for every Λ^λ-open set G containing p,$A \cap G \neq \phi$, since $p \in Cl^{\Lambda^\lambda}(A)$.Since U is Λ^λ-open and $x, p \in U$, $A \cap U \neq \phi$. Hence $x \in Cl^{\Lambda^\lambda}(A)$. Hence $Cl^{\Lambda^\lambda}(A) = Cl^{\Lambda^\lambda}(Cl^{\Lambda^\lambda}(A))$.By (4), $Cl^{\Lambda^\lambda}(A)$ is Λ^λ-closed.

Definition 3.7:A subset A of a topological space (X, τ) is said to be λ-locally closed if $A = S \cap P$, where S is λ-open in X and P is λ-closed in X.

Lemma 3.8: Let A be Λ^λ-closed subset of a topological space (X, τ). Then we have,

1) $A = T \cap Cl^\lambda(A)$, where T is a Λ_*^λ-set.

2) $A = A^{\Lambda_*^\lambda} \cap Cl^\lambda(A)$

Lemma 3.9: A subset $A \subseteq (X, \tau)$ is $\Lambda - g - closed$ iff $Cl^\lambda(A) \subseteq A^{\Lambda_*^\lambda}$.

Proposition3.10: For a subset A of a topological space the following conditions are equivalent.

(1) A is λ - closed.

(2) A is Λ-g-closed and λ-locally closed

(3) A is Λ-g-closed and Λ^λ- closed.

Proof: $(1) \Rightarrow (2)$ Every λ-closed set is both Λ-g-closed and λ-locally closed.

$(2) \Rightarrow (3)$ This is obvious from the fact that every λ-locally closed is a Λ^λ-closed.

$(3) \Rightarrow (1)$ A is Λ-g-closed , so by Lemma 3.9, $Cl^{\lambda}(A) \subseteq A^{\Lambda_*^{\lambda}}$. A is Λ^{λ}-closed, so by Lemma 3.8, $A = A^{\Lambda_*^{\lambda}} \cap Cl^{\lambda}(A)$. Hence $A = Cl^{\lambda}(A)$, i.e., A is λ-closed.

4. Λ^{λ}-closed Maps

In this section, we introduce Λ_*^{λ}- closed maps. Λ_*^{λ}- open maps, Λ^{λ}-closed maps, Λ^{λ}-open maps, $\Lambda^{\lambda*}$-closed maps and $\Lambda^{\lambda*}$-open maps.

Definition 4.1:

A map f : (X,τ)→(Y,σ) is said to be a Λ_*^{λ}-closed map if the image of every closed set in (X,τ) is V_*^{λ}-set in (Y,σ).

Example 4.2:

(a) Let X=Y = {a, b, c, d}, τ = {ϕ, X, {a, b}, {a, b, c}} and

σ = {ϕ, Y, {a},{a, b},{a, c, d}}. Define a map

f : (X,τ)→(Y,σ) by f (a) = c, f (b) =d, f (c) = b and f (d) =a.

Then f is a Λ_*^{λ}-closed.

(b) Let X=Y= {a, b, c, d}, τ = {ϕ, X, {a}, {b}, {a, b}} and

σ = {ϕ, Y, {d}, {b, c, d}}. Let f : (X,τ)→(Y,σ) be the identity map

f ({c, d}) = {c, d} is not a V_*^{λ}-set. Hence f is not a Λ_*^{λ}-closed map.

Definition 4.3: A map f : (X,τ)→(Y,σ) is said to be Λ^{λ}-closed if the image of every closed set in (X,τ) is Λ^{λ}-closed in (Y,σ).

Example 4.4:

(a) Let X=Y = {a, b, c, d, e}, τ = {ϕ, X, {a},{d,e}, {a, d, e}, {b, c, d, e}} and

σ = {ϕ, Y, {b, c}, {b, c, d},{a, b, d, e}}. Define a map f : (X,τ)→(Y,σ) by

f (a) = d, f (b) =e, f (c) = a, f (d) =c and f (e) =b. Then f is a Λ^{λ}-closed map.

(b) Let X=Y= {a, b, c, d}, τ = {ϕ, X,{a, b}, {a, b, c}} and

σ = {ϕ, Y,{a},{b, c}, {a, b, c}, {b, c, d}}. Let f : (X,τ) →(Y,σ) be an identity map. f ({c, d}) = {c, d} is not a Λ^{λ}-closed set. Hence f is not a Λ^{λ}- closed map.

Definition 4.5: A map f : (X,τ)→(Y,σ) is said to be Λ–g-closed if the image of every closed set in (X,τ) is Λ–g-closed in (Y,σ) .

Example 4.6:

(i) Let X=Y = {a, b, c, d}, τ = {ϕ, X ,{d}, {a, d}, {c, d}, {a, c, d}} and

σ = {ϕ ,Y,{a ,b},{c, d}}. Define an identity map f : X→Y is Λ-g-closed.

(ii) Let X=Y={a, b, c, d}, τ = {ϕ ,X,{a}, {a, c}, {a, c, d}} and

σ ={ϕ ,Y,{d},{c, d}, {a, c, d}}. Define a map f : (X,τ)→(Y,σ) by f (a)=d,

f b)=b, f (c)=a, f (d)=c. But {b, c} is not a Λ-g-closed. Hence f is not

Λ-g-closed.

Definition 4.7:

A map f : (X,τ)→(Y,σ) is said to be $\Lambda^{\lambda}-\lambda-$closed if the image of every $\lambda-$closed set in (X,τ) is $\Lambda^{\lambda}-$closed in (Y,σ).

Example 4.8:

The function f which is defined in example 4.2(a) is $\Lambda^{\lambda}-\lambda-$closed.

Proposition 4.9:

Every Λ^{λ}-$\lambda-$closed map is a $\Lambda^{\lambda}-$closed map.

Proof:

Let f : (X, τ)→(Y,σ) be a Λ^{λ}-$\lambda-$closed map. Let B be a closed set in (X,τ) and hence B is a $\lambda-$ closed set in (X, τ). By assumption f (B) is $\Lambda^{\lambda}-$closed in (Y,σ). Hence f is a $\Lambda^{\lambda}-$closed map.

Theorem 4.10:

Let f : (X,τ)→(Y,σ) and g: $(Y,\sigma)\rightarrow(Z,\eta)$ be two mappings such that their composition gof: (X,τ)→(Z,η) be a $\Lambda^{\lambda}-$closed mapping. Then the following statements are true if

1) f is continuous and surjective then g is Λ^{λ}-closed.

2) g is Λ^{λ}-irresolute and injective then f is Λ^{λ} -closed.

3) f is g-continuous, surjective and (X,τ) is a $T_{1/2}-$ space then g is Λ^{λ}-closed

Proof:

1) Let A be a closed set in (Y,σ). Since f is continuous, f^{-1}(A) is closed in (X,τ) and since gof is $\Lambda^{\lambda}-$closed, (gof) (f^{-1}(A)) is $\Lambda^{\lambda}-$closed in (Z,η). i.e., g (A) is Λ^{λ}-closed in (Z,η).

2) Let B be a closed set of (X,τ). Since gof is Λ^{λ} -closed, (gof) (B) is Λ^{λ} -closed in (Z,η). Since g is Λ^{λ} -irresolute $g^{-1}((gof)$ (B)) is Λ^{λ} -closed in (Y,σ).i.e., f (B) is Λ^{λ} –closed in (Y,σ), Since g is injective. Thus f is a Λ^{λ} –closed map.

3) Let A be a closed set of (Y,σ). Since f is g-continuous, f^{-1}(A) is g- closed in (X,τ). Since (X,τ) is a $T_{1/2}$ -space, f^{-1}(A) is closed in (X,τ) and so as in (i), g is a Λ^{λ} –closed map.

Theorem 4.11: Let f : (X,τ) $\rightarrow (Y,\sigma)$ and g: $(Y,\sigma) \rightarrow (Z,\eta)$ be two mappings such that their composition gof: (X,τ) $\rightarrow (Z,\eta)$ be a Λ^{λ} - λ –closed mapping. Then the following is true if

1) f is $\lambda -$irresolute and surjective then g is Λ^{λ} - $\lambda -$closed.

2) g is Λ^{λ} -irresolute and injective then f is Λ^{λ} - λ -closed.

Proof:

1) Let A be a λ -closed set of (Y,σ). Since f is $\lambda -$ irresolute, f^{-1}(A) is λ -closed in (X,τ) and since gof is $\Lambda^{\lambda} - \lambda -$closed, (gof) (f^{-1}(A)) is Λ^{λ} –closed in (Z,η).

i.e., g(A) is Λ^{λ} -closed in (Z,η). Hence g is $\Lambda^{\lambda} - \lambda -$closed.

2)Let B be a λ -closed set of (X,τ). Since gof is Λ^{λ} - $\lambda -$closed, (gof) (B) is Λ^{λ} - closed in (Z,η). Since g is Λ^{λ} -irresolute g^{-1}((gof) (B)) is Λ^{λ} -closed in (Y,σ).i.e., f (B) is

Λ^{λ} –closed in (Y,σ), since g is injective. Thus f is Λ^{λ} - $\lambda -$closed map.

Definition 4.12:

A map $f : (X,\tau) \rightarrow (Y,\sigma)$ is said to be a Λ^{λ} -open map if the image f (A) is Λ^{λ} -open in (Y,σ) for each open set A in (X,τ) .

Proposition: 4.13:

For any bijection $f : (X,\tau) \rightarrow (Y,\sigma)$, the following statements are equivalent.

1) $f^{-1} : (Y,\sigma) \rightarrow$ (X,τ) is Λ^{λ} -continuous.

2) f is a Λ^{λ} -open map and

3) f is a Λ^{λ} -closed map.

Proof:

(1)$\Rightarrow$(2) : Let U be an open set of (X,τ). By assumption (f^{-1})$^{-1}$ (U) = f (U)

is Λ^{λ} -open in (Y,σ) and so f is Λ^{λ} -open.

(2) $\Rightarrow$ (3): Let F be a closed set of (X,τ). Then X-F is open in (X,τ). By assumption, f (X-F) is Λ^{λ}-open in (Y,σ) and therefore f (X-F) =(Y- f (F)) is Λ^{λ}-open in (Y,σ) and therefore f (F) is Λ^{λ}-closed in (Y,σ). Hence f is Λ^{λ}-closed.

(3) $\Rightarrow$ (1) :Let F be a closed set of (X,τ). By assumption, f (F) is Λ^{λ}-closed in (Y,σ). But f (F) = (f^{-1})$^{-1}$(F) and therefore f^{-1} is Λ^{λ}-continuous on Y.

Definition 4.14: A map $f:(X,\tau)\to(Y,\sigma)$ is said to be a $\Lambda^{\lambda}-\lambda$-open map if the image f (A) is Λ^{λ}-open in (Y,σ) for each $\lambda-$open set A in (X,τ).

Proposition: 4.15:

For any bijection $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent.

1) $f^{-1}:(Y,\sigma)\to(X,\tau)$ is $\Lambda^{\lambda}-\lambda$-continuous.

2) f is a $\Lambda^{\lambda}-\lambda$-open map and

3) f is a $\Lambda^{\lambda}-\lambda$-closed map

Proof:

(1)$\Rightarrow$(2) Let U be an λ-open set of (X,τ). By assumption (f^{-1}) $^{-1}$(U) = f (U) is Λ^{λ}-open in (Y,σ) and so f is Λ^{λ}-λ-open.

(2) $\Rightarrow$ (3) Let F be a λ- closed set of (X,τ). Then X- F is λ- open in (X,τ). By assumption, f (X-F) is Λ^{λ}-open in (Y,σ) i.e., f (X-F) = Y- f (F) is Λ^{λ}-open in (Y,σ) and there f (F) is Λ^{λ}–closed in (Y,σ). Hence f is Λ^{λ}–closed.

(3) $\Rightarrow$(1) Let F be a λ–closed set in (X,τ). By assumption. f (F) is Λ^{λ}-λ- closed in (Y,σ) . But f(F) = (f^{-1})$^{-1}$ (F) and therefore f^{-1} is Λ^{λ}-λ- continuous on Y.

Definition 4.16: (i) A map $f:(X,\tau)\to(Y,\sigma)$ is said to be a λ^{*}-closed map if the image of f (A) is λ-closed in (Y,σ) for every λ-closed set A in (X,τ).

(ii) A map $f:(X,\tau)\to(Y,\sigma)$ is said to be a $\Lambda^{\lambda *}$–closed map if the image f (A) is Λ^{λ}–closed in (Y,σ) for every Λ^{λ}–closed set A in (X,τ).

(iii) A map $f:(X,\tau)\to(Y,\sigma)$ is said to be a $\Lambda-gc$–closed if the image f (A) is $\Lambda-g$–closed in (Y,σ) for every $\Lambda-g$–closed set A in (X,τ).

Example: 4.17

(i) Let $X=Y=\{a,b,c,d,e\},\tau=\{\phi,X,\{a\},\{a,b,c\}\}$ and $\sigma=\{\phi,Y,\{a\},\{b\},\{a,b\}\}$. Define a map $f:(X,\tau)\to(Y,\sigma)$ by $f(a)=a,\ f(b)=c,\ f(c)=d,\ f(d)=b$ and $f(e)=e.$ Then f is λ^{*}–closed as well as $\Lambda^{\lambda*}$–closed.

Remark 4.18:

Since every closed set is a Λ^{λ}–closed set we have every $\Lambda^{\lambda*}$ – closed map is a Λ^{λ}–closed map.

The converse is not true in general as seen from the following example.

Example 4.19:

(i) Let $X=Y=\{a,b,c,d\},\tau=\{\phi,X,\{a\},\{a,b,c\}\}$ and $\sigma=\{\phi,Y,\{a\},\{b\},\{a,b\}\}$. Defineamap $f:(X,\tau)\to(Y,\sigma)$ by $f(a)=a,\ f(b)=b,\ f(c)=d,\ f(d)=c$. Then f is Λ^{λ}–closed but not $\Lambda^{\lambda^{*}}$ –closed, because for the Λ^{λ}–closed set {d} in (X,τ) $f(\{d\})=c$ which is not a Λ^{λ}–closed set in (Y,σ).

Proposition 4.20: For any bijection $f:(X,\tau)\to(Y,\sigma)$ the following are equivalent:

(i) $f^{-1}:(Y,\sigma)\to(X,\tau)$ is Λ^{λ}-irresolute

(ii) f is a Λ^{λ} c-open map and

(iii) f is a Λ^{λ} c-closed map.

Proof: Similar to proposition 4.13.

Definition 4.21:

Let A be a subset of X. A mapping $r:X\to A$ is called a Λ^{λ}– continuous retraction if r is a Λ^{λ}-continuous and the restriction if r is a Λ^{λ}– continuous and the restriction r_A is the identity mapping on A.

Definition 4.22: A topological space (X,τ) is called a Λ^{λ}– Hausdorff if for each pair x,y of distinct points of X there exists Λ^{λ}-neighborhoods U_1 and U_2 of x and y, respectively, that are disjoint.

Theorem 4.24:

Let A be a subset of X and $r : X \to A$ be a Λ^{λ}-continuous retraction. If X is Λ^{λ}-Hausdorff, then A is a Λ^{λ}-closed set of X.

Proof:

Suppose that A is not Λ^{λ}-closed. Then there exists a point x in X such that $x \in c\ell^{\Lambda^{\lambda}}(A)$ but $x \notin A$. It follows that $r(x) \neq x$ because r is Λ^{λ}–continuous retraction. Since X is Λ^{λ}-Hausdroff, there exists disjoint Λ^{λ}-open sets U and V in X such that $x \in U$ and $r(x) \in V$. Now let W be an arbitrary Λ^{λ}-neighborhood of x. Then $W \cap U$ is a Λ^{λ}-neighborhood of x. Since $x \in C\ell^{\Lambda^{\lambda}}(A)$, we have $(W \cap U) \cap A \neq \phi$. Therefore there exists a point y in $W \cap U \cap A$. Since $y \in A$, We have $r(y) = y \in U$ and hence $r(y) \notin V$. This implies that $r(W) \not\subset V$ because $y \in W$. This is contrary to the Λ^{λ}–continuity of r. Consequently, A is a Λ^{λ}-closed set of X.

Theorem 4.25

Let $\{Xi : i \in I\}$ be any family of topological space. If $f : X \to \pi xi$ is a Λ^{λ}–continuous mapping, then P_{r_i} of : $X \to Xi$ is Λ^{λ}-continuous for each $i \in I$, where P_{r_i} is the projection of πx_j on to X_i.

Proof: We shall consider a fixed $i \in I$. Suppose U_i is an arbitrary open set in X. Then $P_{ri}^{-1}(U_i)$ is open in πXi. Since f is Λ^{λ}-continuous, we have $f^{-1}(p_{r_i}{}^{-1}(U_i)) = (p_{r_i} \circ f)^{-1}(U_i)$ Λ^{λ}-open in X. Therefore $P_{r_i} \circ f$ is Λ^{λ}-continuous.

Proposition 4.26:

A mapping $f : (X, \tau) \to (Y, \sigma)$ is Λ^{λ}-closed if and only if $Cl^{\Lambda^{\lambda}}(f(A)) \subset f(Cl(A))$ for every subset A of (X, τ).

Proof: Suppose that *f* is Λ^{λ}-closed and $A \subset X$. Then *f*(Cl(A)) is Λ^{λ}-closed in (Y, σ). We have $f(A) \subset f(Cl(A))$ and by proposition 3.6, $Cl^{\Lambda^{\lambda}}(f(A)) \subset Cl^{\Lambda^{\lambda}}(f(Cl(A))) = f(Cl(A))$.Conversely, let A be any closed set in (X, τ). By hypothesis and proposition 3.6, we have A=Cl(A) and so $f(A) = f(Cl(A)) \supset Cl^{\Lambda^{\lambda}}(f(A))$. i.e., *f*(A) is Λ^{λ}-closed and hence *f* is Λ^{λ}-closed.

Theorem 4.27:

A mapping $f:(X,\tau)\rightarrow(Y,\sigma)$ is λ-closed if and only if *f* is both Λ-g-closed and Λ^{λ}-closed.

Proof: Let V be a closed set in (X,τ). As *f* is λ-closed, *f*(V) is a λ-closed in (Y,σ). By Proposition 3.10, *f*(V) is a Λ-g-closed and Λ^{λ}-closed set. Hence *f* is Λ-g-closed and Λ^{λ}-closed.

Conversely, let V be closed in (X,τ).As *f* is Λ-g-closed and Λ^{λ}-closed *f*(V) is both Λ-g-closed and Λ^{λ}-closed set. Hence *f*(V) is λ-closed by Proposition 3.10.

5. Λ^{λ}-Homeomorphisms.

In this section we introduce and study two new homeomorphisms namely Λ^{λ}-homeomorphism and Λ^{λ^*} homeomorphism.

Definition 5.1:

A bijection $f:(X,\tau)\rightarrow(y,\sigma)$ is called λ-homeomorphism if f is both

λ-continuous and λ-open.

Proposition 5.2: Every homeomorphism is a λ-homeomorphism.

Proof: Follows from definitions.

The converse of the Proposition 5.2 need not be true as see from the following example.

Example 5.3:

Let $X=Y=\{a,b,c,d\},\tau=\{\phi,X,\{a\},\{a,b,c\}\}$ and $\sigma=\{\phi,Y,\{a\},\{b\},\{a,b\}\}$. Define a map $f:(X,\tau)\rightarrow(y,\sigma)$ by $f(a)=a, f(b)=c, f(c)=d$ and $f(d)=b.$ Then f is λ- homeomorphism but not a homeomorphism, because it is not continuous.

Thus, the class of λ-homeomorphisms properly contains the class of homeomorphism.

Definition 5.4: A bijection $f:(X,\tau)\rightarrow(Y,\sigma)$ is called Λ^{λ}–homeomorphism if f is both Λ^{λ}-continuous and Λ^{λ}-open.

Proposition 5.5: Every λ–homeomorphism is a Λ^{λ}–homeomorphism

proof: Follows from definitions.

The converse of the Proposition 5.5 need not be true as seen from the following example.

Example5.6:

Let $X=Y=\{a,b,c,d\},\tau=\{\phi,X,\{a,b\},\{a,b,c\}\}$ and $\sigma=\{\phi,Y,\{a\},\{a,b\},\{a,c,d\}\}$. Define a map $f:(X,\tau)\rightarrow(Y,\sigma)$ by $f(a)=c, f(b)=d, f(c)=a$ and $f(d)=b.$ Then f is Λ^{λ}–homeomorphism but not a homeomorphism. Because it is not a λ-continuous function.

Thus, the class of Λ^{λ}-homeomorphisms properly contains the class of λ-homeomorphisms.

Definition 5.7:

A bijection $f:(X,\tau)\to(Y,\sigma)$ is called $\Lambda^{\lambda}-\lambda-$homeomorphism if f is both $\Lambda^{\lambda}-\lambda-$continuous and $\Lambda^{\lambda}-\lambda-$open.

Proposition 5.8:

Every $\Lambda^{\lambda}-\lambda-$homeomorphism is a $\Lambda^{\lambda}-$homeomorphism but not conversely.

Proof: Follows from definitions.

The converse of the Proposition 5.8 need not be true as seen from the following example.

Example 5.9:

The function f in 5.3 is $\Lambda^{\lambda}-$homeomorphism but not $\Lambda^{\lambda}-\lambda-$homeomorphism. Because f is not $\Lambda^{\lambda}-\lambda-$continuous.

Thus, the class of $\Lambda^{\lambda}-$homeomorphisms property contains the class of $\Lambda^{\lambda}-\lambda-$homeomorphisms.

Proposition 5.10: Let $f:(X,\tau)\to(Y,\sigma)$ be a bijection $\Lambda^{\lambda}-$continuous map. Then the following statements are equivalent:

(i) f is a Λ^{λ}-open map

(ii) f is a Λ^{λ}-homeomorphism

(iii) f is a Λ^{λ}-closed map.

Proof: (i) $\Leftrightarrow$ (ii) Follows from the definition.

(i) $\Leftrightarrow$ (iii) Follows from proposition 3.13.

Proposition 5.11: Let $f:(X,\tau)\to(Y,\sigma)$ be a bijection $\Lambda^{\lambda}-\lambda-$continuous map. Then the following statements are equivalent.

(i) f is a $\Lambda^{\lambda}-\lambda-$-open map

(ii) f is a $\Lambda^{\lambda}-\lambda-$-homeomorphism

(iii) f is a $\Lambda^{\lambda}-\lambda-$-closed map.

Proof: (i) $\Leftrightarrow$ (ii) Follows from the definition.

(i) $\Leftrightarrow$ (iii) Follows from proposition 3.15.

The composition of two $\Lambda^{\lambda}-$homeomorphism maps need not be a $\Lambda^{\lambda}-$homeomorphism as can be seen from the following example.

Example 5.12:

Let $X = Y = Z = \{a,b,c,d,e\}, \tau = \{\phi, X, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$, $\sigma = \{\phi, Y, \{c,d\}, \{b,c,d\}, \{a,c,d\}\}$

and $\eta = \{\phi, Z, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$, Define a map $f:(X,\tau) \to (Y,\sigma)$ by $f(a) = e, f(b) = b, f(c) = a, f(d) = c, f(e) = d$ and $g:(Y,\sigma) \to (Z,\eta)$ by $g(a) = a, g(b) = b, g(c) = e, g(d) = d, g(e) = c.$ Then f and g are Λ^{λ}-homeomorphisms but their composition $gof:(X,\tau) \to (Z,\eta)$ is not a Λ^{λ}-homeomorphism, because $(gof)^{-1}(\{a,b\}) = \{b,c\}$ which is not a Λ^{λ}-open set in (z,η). Therefore gof is not a Λ^{λ}-continuous map and so gof is not a Λ^{λ}-homeomorphism.

We next introduce a new class of maps called $\Lambda^{\lambda *}$-homeomorphisms which forms a sub class of Λ^{λ}-homeomorphisms. This class of maps is closed under composition of maps.

Definition 5.13:

A bijection $f:(X,\tau) \to (Y,\sigma)$ is said to be λ^{*}-homeomorphism if both

f and f^{-1} are λ-irresolute.

We denote the family of all λ-homeomorphisms (resp. λ^{*}-homeomorphism and homeomorphism) of a topological space (X,τ) on to itself by $\lambda - h(X,\tau)$. (resp. $\lambda^{*} - h(X,\tau)$ and $h(X,\tau)$.

Proposition 5.14:

Every λ^{*}-homeomorphism is a λ-homeomorphism but not conversely (i.e) for any space (X,τ), $\lambda^{*} - h(X,\tau) \subset \lambda - h(X,\tau)$.

Proof:

It follows from the fact that every λ-irresolute map is a λ -continuous map and the fact that λ^{*}-open map is λ-open map.

The function f in example 5.3 is a $\lambda -$homeomorphism but not a λ^{*}-homeomorphism, since for the $\lambda -$closed set $\{a,b\}$ in (Y,σ), $f^{-1}(\{a,b\}) = \{a,d\}$ which is not $\lambda -$closed in (X,τ). Therefore f is not $\lambda -$irresolute and so f is not a λ^{*}-homeomorphism.

Definition 5.15:

A bijection $f:(X,\tau) \to (Y,\sigma)$ is said to be $\Lambda^{\lambda *}$-homeomorphism if both f and f^{-1} are Λ^{λ}-irresolute.

We denote the family of all Λ^{λ}-homeomorphisms (resp. $\Lambda^{\lambda*}$-homeomorphism) of a topological space (X,τ) on to itself by $\lambda^{*}-h(X,\tau)$. (resp. $\Lambda^{\lambda^{*}}-h(X,\tau)$.

Proposition 5.16:

Every $\Lambda^{\lambda*}$-homeomorphism is a Λ^{λ}-homeomorphism but not conversely (i.e) for any space (X,τ), $\Lambda^{\lambda^{*}}-h(X,\tau)\subset\Lambda^{\lambda}-h(X,\tau)$.

Proof:

Follows from theorem 3.5 and the fact that every $\Lambda^{\lambda*}$-open map is Λ^{λ}-open. The function f in Example 5.4 is a Λ^{λ}-homeomorphism but not a $\Lambda^{\lambda*}$-homeomorphism, since for the Λ^{λ}-closed set $\{a,b.e\}$ in (y,σ), $f^{-1}(\{a,b,e\})=\{a,b,c\}$ which is not Λ^{λ}-closed set in (X,τ). Therefore f is not Λ^{λ}-irresolute and so f is not a $\Lambda^{\lambda*}$-homeomorphism.

Theorem 5.17:

If $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ are $\Lambda^{\lambda*}$-homeomorphism, then their composition $gof:(X,\tau)\to(Z,\eta)$ is also $\Lambda^{\lambda*}$-homeomorphisms.

Proof:

Let B be a Λ^{λ}-open set in (Z,η). Now, $(gof)^{-1}(B)=f^{-1}(g^{-1}(B)=f^{-1}(C)$, where $C=g^{-1}(B)$. By hypothesis, C is Λ^{λ}-open in (Y,σ) and so again by hypothesis, $f^{-1}(C)$ is Λ^{λ}-open in (X,τ). Therefore gof is Λ^{λ}-irresolute. Also for a Λ^{λ}-open set G in (X,τ). We have $(gof)(G)=g(f(G))=g(V)$, where $V=f(G)$. By hypothesis $f(G)$ is Λ^{λ}-open in (Y,σ) and so again by hypothesis, $g(f(G))$ is Λ^{λ}-open in (Z,η) .i.e., $(gof)(G)$ is Λ^{λ}-open in (Z,η) and therefore $(gof)^{-1}$ is Λ^{λ}-irresolute. Hence gof is a $\Lambda^{\lambda*}$-homeomorphism.

Theorem 5.18:

The set $\Lambda^{\lambda*}-h(X,\tau)$ is a group under the composition of maps.

Proof:

Define a binary operation *: $\Lambda^{\lambda*}-h(X,\tau)\ X\ \Lambda^{\lambda*}-h(X,\tau)\to\Lambda^{\lambda*}-h(X,\tau)$ by $f*g=gof$ for all $f,g\in\Lambda^{\lambda*}-h(X,\tau)$ and o is the usual operation of composition of maps. Then by theorem 5.17, $gof\in\Lambda^{\lambda*}-h(X,\tau)$. We know that the composition of maps is associative and the identity map $I:(X,\tau)\to(X,\tau)$ belonging to $\Lambda^{\lambda*}-h(X,\tau)$ serves as the identity element. If

$f \in \Lambda^{\lambda*} - h(X,\tau)$, then $f^{-1}\Lambda^{\lambda*} - h(X,\tau)$ such that $fof^{-1} = f^{-1}of = I$ and so inverse $(\Lambda^{\lambda*} - h(X,\tau), o)$ is a group under the operation of composition of maps.

Theorem 5.19:

Let $f:(X,\tau) \to (Y,\sigma)$ be a $\Lambda^{\lambda*}$- homeomorphism. Then f induces an isomorphism from the group $\Lambda^{\lambda*} - h(X,\tau)$ on to the group $\Lambda^{\lambda*} - h(Y,\sigma)$.

Proof:

Using the map g, we define a map $\wp_g : \Lambda^{\lambda*} - h(X,\tau) \to \Lambda^{\lambda*} - h(Y,\sigma)$ by

$\wp_g(h) = gohog^{-1}$ for every $h \in \Lambda^{\lambda*} - h(X,\tau)$. Then $\wp_g$ is a bijection. Further, for all $h_1, h_2 \in \Lambda^{\lambda*} - h(X,\tau)$, $\wp_g(h_1oh_2) = go(h_1oh_2)og^{-1} = (goh_1og^{-1})o(goh_2og^{-1}) = \wp_g(h_1)o\wp_g(h_2)$.

Therefore, $\wp_g$ is a homeomorphism and so it is an isomorphism induced by g.

Theorem 5.20:

$\Lambda^{\lambda*}$-homeomorphisms is an equivalence relation in the collection of all topological spaces.

Proof: Reflexivity and symmetry are immediate and transitivity follows from Theorem 5.17.

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