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The Relationships Between the Bernoulli Numbers B_{2n} & a_n And the Euler Numbers E_{2n} & b_n

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Abstract

Traditionally, the values of Riemann zeta function at the even positive integers have been formulated in terms of Bernoulli numbers B_{2n} and the sums of the alternating series of odd powers of the reciprocal of odd positive integers have been calculated in terms of Euler numbers E_{2n}. However, the present author reproduced the sum of the same series using different procedures in terms of two rational numbers a_n and b_n. In this paper, the relationships between B_{2n} & a_n and E_{2n} & b_n have been established. Consequently, some of the theorems and corollaries in [5] have been restated in terms of B_{2n} and E_{2n}.

Key Words: Riemann zeta function, Bernoulli numbers, Euler numbers, Sum of Series.

1. Introduction

When the present author worked on the paper [5] in 2010, he was not aware of the works that have produced similar results in terms Bernoulli and Euler numbers earlier [3, 4]. One of the friends of the author first told him about existence of such works in terms of B_{2n} & E_{2n} and suggested him to find the possible connections between B_{2n} & a_n and E_{2n} & b_n. When investigated further, author found some other people did similar works in the recent years [8, 9, 10]. However, the approach of [5] was different from others. In section 2, the connection between B_{2n} & a_n has been established and in order to do it, the formula for calculating the values of Riemann zeta function at the positive even integers in terms of Bernoulli's numbers B_{2n} has been derived. For finding the connection between E_{2n} & b_n, the derivation of computing the sum of alternating series of odd powers of the reciprocals of odd positive integers in terms of the Euler's numbers E_{2n} has been shown in section 3. Acknowledgement and the References are provided respectively in section 4 and 5.

2. Relation Between B_{2n} and a_n

Derivation of

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}}{2\cdot(2n)!}B_{2n},\ n \geq 1,$$ where B_n are the Bernoulli's numbers.

The Riemann zeta function ζ [2] is defined by the formula

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i}\int_{-\infty}^{\infty}\frac{(-x)^s}{e^x - 1}\cdot\frac{dx}{x}, \tag{1a}$$

where $\Pi(s) = \int_0^{\infty} e^{-x}x^s dx\ (s > -1)$ and for real values of s greater than one, $\zeta(s)$ is equal to the Dirichlet's function

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}. \tag{1b}$$

However, formula (1a) for $\zeta(s)$ is valid for all s. In fact, since the integral in (1a) clearly converges for all values of s, real or complex, and since the function it defines is complex analytic, the function $\zeta(s)$ of (1a) is defined and analytic at all points with the possible exception of the points $s = 1, 2, 3, \cdots$, where $\Pi(-s)$ has poles. For $s = 2, 3, 4, \cdots$, formula (1b) shows that $\zeta(s)$ has no poles, and hence the integral in (1a) must have a zero which cancels the poles of $\Pi(-s)$ at these points, a fact which also follows immediately from Cauchy's theorem. At $s = 1$ formulas (1b) shows that $\zeta(s) = \infty$ as $s \to 1$, hence $\zeta(s)$ has a simple (since the pole of $\Pi(-s)$ is simple) pole $s = 1$. Thus formula (1a) defines a function $\zeta(s)$ which is analytic at all points of the complex $s-$plane except for a simple pole at $s = 1$. This function coincides with (1b) for real values of $s > 1$ and in fact, by analytic continuation, throughout the half plane Re $s > 1$.

Let $f(x) = \dfrac{x}{e^x - 1} = B_0 + \dfrac{B_1}{1!}x + \dfrac{B_2}{2!}x^2 + \dfrac{B_3}{3!}x^3 + \cdots + \dfrac{B_n}{n!}x^n + \cdots,$ (2)

where $B_0 = \lim\limits_{x\to 0}\dfrac{x}{e^x - 1} = \lim\limits_{x\to 0}\dfrac{x}{1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots - 1} = \lim\limits_{x\to 0}\dfrac{1}{1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \cdots} = 1$ and the series

converges on the disk $D: |x| < 2\pi$. Moreover, the element $\{D, f(x)\}$ has singular point at

$x = \pm 2\pi i$, since $\lim\limits_{x\to \pm 2\pi i}\dfrac{x}{e^x - 1} = \infty$, and hence (2) has radius of convergence 2π.

Then $f(-x) = \dfrac{-x}{e^{-x} - 1} = \dfrac{-xe^x}{(e^{-x} - 1)e^x} = \dfrac{xe^x}{e^x - 1}, \ |x| < 2\pi$

$$= B_0 - \frac{B_1}{1!}x + \frac{B_2}{2!}x^2 - \frac{B_3}{3!}x^3 + \cdots + (-1)^n\frac{B_n}{n!}x^n + \cdots. \quad (3)$$

(replacing x by $(-x)$ in (2)). Subtracting (3) from (2) gives

$$f(x) - f(-x) = \frac{x}{e^x - 1} - \frac{xe^x}{e^x - 1} = \frac{-x(e^x - 1)}{e^x - 1} = -x$$

$$= 2\frac{B_1}{1!}x + 2\frac{B_3}{3!}x^3 + \cdots + 2\frac{B_{2n+1}}{(2n+1)!}x^{2n+1} + \cdots \quad (4)$$

Equating the coefficients of the like terms one obtains,

$$2B_1 = -1, \ B_3 = B_5 = \cdots = B_{2n+1} = \cdots = 0.$$

Consequently, $f(x)=\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}x^{2n}, |x|<2\pi.$ (5)

One can write, $\cot x=\frac{\cos x}{\sin x}=i\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}}=i\frac{e^{2ix}+1}{e^{2ix}-1}=i+\frac{2i}{e^{2ix}-1}$ is analytic for $|x|<\pi$ and then

$x\cot x=ix+\frac{2ix}{e^{2ix}-1}=ix+1-\frac{2ix}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(2ix)^{2n}=1+\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}B_{2n}}{(2n)!}x^{2n}$ (replacing x by $2xi$ in (5)).

$$\Rightarrow \cot x-\frac{1}{x}=\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n}B_{2n}}{(2n)!}x^{2n-1} \quad (6)$$

Again, one can write [6],

$$\cot x-\frac{1}{x}=\sum_{k=1}^{\infty}\left(\frac{1}{x-k\pi}+\frac{1}{x+k\pi}\right), |x|<\pi. \quad (7)$$

However, $\frac{1}{x-k\pi}+\frac{1}{x+k\pi}=-\sum_{m=0}^{\infty}\frac{x^m}{(k\pi)^{m+1}}+\sum_{m=1}^{\infty}(-1)^m\frac{x^m}{(k\pi)^{m+1}}=-2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(k\pi)^{2n}}, |x|<k\pi.$ (8)

The coefficients of even powers of x in the Taylor series of $\cot x-\frac{1}{x}$ vanish, since $\cot x-\frac{1}{x}$ is an odd function. The coefficients of x^{2n-1} in (8) can be written as

$$-2\sum_{n=1}^{\infty}\frac{1}{(k\pi)^{2n}}=-\frac{2}{\pi^{2n}}\sum_{n=1}^{\infty}\frac{1}{k^{2n}}.$$

That converts (7) to

$$\cot x-\frac{1}{x}=\sum_{n=1}^{\infty}\left[-\frac{2}{\pi^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}}\right]x^{2n-1}. \quad (9)$$

Comparing (6) and (9) one can write,

$$\frac{2}{\pi^{2n}}\sum_{n=1}^{\infty}\frac{1}{k^{2n}}=(-1)^{n+1}\frac{2^{2n}B_{2n}}{(2n)!}$$

$$\Rightarrow \zeta(2n)=\sum_{n=1}^{\infty}\frac{1}{k^{2n}}=(-1)^{n+1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}, \quad (10)$$

which had been found by Euler [3].

In [5], **Corollary 1.2**, the following relationship has been established

$$\zeta(2n)=\frac{2^{2n}}{2^{2n}-1}\pi^{2n}a_0(n), \quad n=1,2,3,\cdots. \quad (11)$$

where a_0 is a rational number (not fixed) that depends on n. Equating (10) and (11) and solving for $a_0(n)$, one can show,

$$a_0(n) = (-1)^{n+1}\frac{2^{2n}-1}{2(2n)!}B_{2n}. \tag{12}$$

This establishes the relationship between B_{2n} and $a_0(n)$.

Proof of Corollary 1.2:

The Riemann zeta can be written as

$$\zeta(n) = \frac{2^2}{2^n-1}j = \sum_{j=0}^{\infty}\frac{1}{(2j+1)^n}.$$

Replacing n by $2n$ in the above relation and using the first part of the **Corollary1.1** which is given by

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k}} = a_0\,\pi^{2k}$$

one can prove the Corollary 1.2.

Remark: Although, in establishing the Theorem 1.1 in [5], especially, the first equation of (10) in [5], the numbers $a_n, n \geq 0,$ have been utilized, for a given value of $n,$ all the numbers $a_n, a_{n-1}, \cdots, a_1$ are already known and only a_0 needs to be computed. Moreover, only a_0 plays a significant role in computing the values of the Riemann zeta function at the even positive integers.

Examples: 1. For $n=1,\ a_0(1) = \frac{3}{2\cdot 2}\cdot\frac{1}{6} = \frac{1}{8},\ \left(B_2 = \frac{1}{6}\right).$

2. For $n=2,\ a_0(2) = (-1)\frac{15}{2\cdot 24}\cdot\left(-\frac{1}{30}\right) = \frac{1}{96},\ \left(B_4 = -\frac{1}{30}\right).$

3. For $n=3,\ a_0(3) = \frac{2^6-1}{2\cdot 6!}\cdot\left(\frac{1}{42}\right) = \frac{1}{960},\ \left(B_6 = \frac{1}{42}\right).$

Using the relation in (12) one can restate the first part of the **Corollary 1.1** and **Corollary 1.2** in [5] respectively as:

Corollary 1.1. For $n \geq 1,\ \sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n}} = a_0(n)\pi^{2n} = (-1)^{n+1}\frac{2^{2n}-1}{2(2n)!}\pi^{2n}B_{2n}.$

Corollary 1.2. For $n \geq 1, \zeta(2n) = \frac{2^{2n}}{2^{2n}-1}\pi^{2n}a_0(n) = (-1)^{n+1}\frac{2^{2n}-1}{2(2n)!}\pi^{2n}B_{2n}$.

3. Relation Between E_{2n} and b_n

Derivation of

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}} = (-1)^n\frac{\pi^{2n+1}}{4^{n+1}(2n)!}E_{2n},\ (n=0,1,2,\cdots),$$ where E_n are the Euler's numbers.

The function

$\sec x = \frac{1}{\cos x}$ is analytic for $|x| < \frac{\pi}{2}$ and therefore $\sec x$ has Maclaurin series representation

$$\sec x = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots.$$

Since $\sec x$ is an even function, all the coefficients of odd powers of x vanish, and hence

$$\sec x = \frac{1}{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots}$$

$$= c_0 + c_2x^2 + c_4x^4 + \cdots + c_{2n}x^{2n} + \cdots,$$

where $c_{2n} = (-1)^n\frac{E_{2n}}{(2n)!}$ and E_n are the Euler's numbers. Then

$$\sec x = \sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{(2n)!}x^{2n},\ |x| < \frac{\pi}{2}. \tag{13}$$

It can be shown [6] that $\sec x$ has the partial fraction decomposition

$$\sec x = \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)\pi}{x^2-\left(k-\frac{1}{2}\right)^2\pi^2} = \sum_{k=1}^{\infty}f_k(x),$$ where

$$f_k(x) = (-1)^k \left[\frac{1}{x - \left(k - \frac{1}{2}\right)\pi} - \frac{1}{x + \left(k - \frac{1}{2}\right)\pi} \right]$$

$$= -(-1)^k \left[\sum_{m=0}^{\infty} \frac{x^m}{\left[\left(k - \frac{1}{2}\right)\pi\right]^{m+1}} + \frac{(-1)^m x^m}{\left[\left(k - \frac{1}{2}\right)\pi\right]^{m+1}} \right]$$

$$= 2(-1)^{k-1} \sum_{n=0}^{\infty} \frac{x^{2n}}{\left[\left(k - \frac{1}{2}\right)\pi\right]^{2n+1}}, \ |x| < \left(k - \frac{1}{2}\right)\pi. \tag{14}$$

The coefficients x^{2n} can be written as series $\frac{2(-1)^{k-1}}{\left[\left(k - \frac{1}{2}\right)\pi\right]^{2n+1}} = 2\left(\frac{2}{\pi}\right)^{2n+1} \frac{(-1)^{k-1}}{(2k-1)^{2n+1}} \ (n = 0, 1, 2, \cdots).$

It follows that

$$\sec x = \sum_{n=0}^{\infty} \left[2\left(\frac{2}{\pi}\right)^{2n+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \right] x^{2n} \quad \left(|x| < \frac{\pi}{2}\right). \tag{15}$$

In (15) the index $(k-1)$ has been replaced by k. Comparing (13) and (15) one can, after some simplification, write

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = (-1)^n \frac{\pi^{2n+1}}{4^{n+1} \cdot (2n)!} E_{2n}, \ (n = 0, 1, 2, \cdots). \tag{16}$$

Euler predicted such a formula and he calculated the sum of the series for $n = 1$ in 1774 [1, 4, 7].

The second part of the **Corollary 1.1** in [5] has the relation

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = b_{2n}\pi\left(\frac{\pi}{2}\right)^{2n} + \sum_{i=1}^{n} b_{2n-2i+1}\pi^{2i}\left(\frac{\pi}{2}\right)^{2n-2i+1} \quad (n = 0, 1, 2, \cdots). \tag{17}$$

Equating (16) and (17) one can write,

$$b_{2n}\pi\left(\frac{\pi}{2}\right)^{2n} + \sum_{i=1}^{n} b_{2n-2i+1}\pi^{2i}\left(\frac{\pi}{2}\right)^{2n-2i+1} = (-1)^n \frac{\pi^{2n+1}}{4^{n+1} \cdot (2n)!} E_{2n}, \ (n = 0, 1, 2, \cdots). \tag{18}$$

This establishes the relationship between B_{2n} and b_n.

Proof of Corollary 1.1:

One can easily prove the part of the Corollary 1.1 by simply substituting $\theta = \frac{\pi}{2}$ in the first and the second part of the Theorem 1.1 below and using the relation

$$\sum_{n=0}^{\infty} \frac{\sin(2n+1)\theta}{2n+1} = \frac{\pi}{4}. \qquad (18a)$$

Here for $n = 0, 1,$ the equality of two sides of (18) has been shown.

For $n = 0$, equation (18) is satisfied: $b_0 \pi = \frac{\pi}{4} E_0 \Rightarrow \frac{\pi}{4} = \frac{\pi}{4} \left(b_0 = \frac{1}{4}, E_0 = 1 \right)$.

Similarly, $n = 1$, the two sides of (18) become equal.

$$\frac{\pi^3}{4} b_2 + \frac{\pi^3}{2} b_1 = (-1) \frac{\pi^3 E_2}{4^2 \cdot 2!}$$
$$\Rightarrow \frac{\pi^3}{4} \cdot \left(-\frac{1}{8} \right) + \frac{\pi^3}{2} \cdot \left(\frac{1}{8} \right) = (-1) \frac{\pi^3}{32} (-1) \quad \left(b_2 = -\frac{1}{8},\ b_1 = \frac{1}{8},\ E_2 = 1 \right)$$
$$\Rightarrow \frac{\pi^3}{32} = \frac{\pi^3}{32}$$

Using the relation in (18), the second part of the **Corollary 1.1** in [5] can be restated as:

Corollary1.1. For $n \geq 1$, $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = b_{2n} \pi \left(\frac{\pi}{2} \right)^{2n} + \sum_{i=1}^{n} b_{2n-2i+1} \pi^{2i} \left(\frac{\pi}{2} \right)^{2n-2i+1} \quad (n = 0, 1, 2, \cdots)$, where

$$b_{2n} \pi \left(\frac{\pi}{2} \right)^{2n} + \sum_{i=1}^{n} b_{2n-2i+1} \pi^{2i} \left(\frac{\pi}{2} \right)^{2n-2i+1} = (-1)^n \frac{\pi^{2n+1}}{4^{n+1} \cdot (2n)!} E_{2n}, \ (n = 0, 1, 2, \cdots).$$

Finally, the **Theorem 1.1.** in [5] is restated here:

Theorem 1.1. For $k \geq 0$,

$$\sum_{n=0}^{\infty} \frac{\cos(2n+1)\theta}{(2n+1)^{2k}} = a_{2k-1} \pi \theta^{2k-1} + \sum_{i=1}^{k} a_{2k-2i} \pi^{2i} \theta^{2k-2i} \quad and$$
$$\sum_{n=0}^{\infty} \frac{\sin(2n+1)\theta}{(2n+1)^{2k+1}} = b_{2k} \pi \theta^{2k} + \sum_{i=1}^{k} b_{2k-2i+1} \theta^{2k-2i+1}, \qquad (19)$$

where the rational numbers a_n and b_n are expressed in term of B_{2n} and E_{2n} respectively in (12) and (18).

Proof of Theorem 1.1:

Here Mathematical Induction is being used for the proof of this theorem. For $k=1$ and $\theta=0$, the first statement of Theorem 1.1 is true since

$$\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{(2n+1)^2}=a_1\pi\theta+a_0\pi^2, \quad \left(a_0=\frac{1}{8}\right)$$

$\Rightarrow 1^2+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{1}{8}\pi^2$ which is a well known result.

It is assumed that it is true for $k=r$, where r is a positive integer. Then

$$\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{(2n+1)^{2r}}=a_{2r-1}\pi\theta^{2r-1}+\sum_{i=1}^{r}a_{2r-2i}\pi^{2i}\theta^{2r-2i}. \tag{20}$$

The proof would be over in one can show that above statement is true for $k=r+1$, that is,

$$\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{(2n+1)^{2(r+1)}}=a_{2r+1}\pi\theta^{2r+1}+\sum_{i=1}^{r+1}a_{2r-2i+2}\pi^{2i}\theta^{2r-2i+2}. \tag{21}$$

Integration of (20) with respect to θ provides

$$\sum_{n=0}^{\infty}\frac{\sin(2n+1)\theta}{(2n+1)^{2r+1}}+c=\frac{a_{2r-1}}{2r}\pi\theta^{2r}+\sum_{i=1}^{r}\frac{a_{2r-2i}}{(2r-2i+1)}\pi^{2i}\theta^{2r-2i+1}. \tag{22}$$

Substitution of $\theta=0$ shows that $c=0$, which turns (22) into

$$\sum_{n=0}^{\infty}\frac{\sin(2n+1)\theta}{(2n+1)^{2r+1}}=\frac{a_{2r-1}}{2r}\pi\theta^{2r}+\sum_{i=1}^{r}\frac{a_{2r-2i}}{(2r-2i+1)}\pi^{2i}\theta^{2r-2i+1}. \tag{23}$$

Integration of (23) with respect to θ produces the equation

$$-\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{(2n+1)^{2r+2}}+c=\frac{a_{2r-1}}{2r(2r+1)}\pi\theta^{2r+1}+\sum_{i=1}^{r}\frac{a_{2r-2i}}{(2r-2i+1)(2r-2i+2)}\pi^{2i}\theta^{2r-2i+2}. \tag{24}$$

Setting $\theta=0$ gives $c=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2(r+1)}}=a_0\pi^{2(r+1)}$ after repeated integration of (18a). Then (24) becomes

$$\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{(2n+1)^{2r+2}}=a_0\pi^{2(r+1)}-\frac{a_{2r-1}}{2r(2r+1)}\pi\theta^{2r}-\sum_{i=1}^{r}\frac{a_{2r-2i}}{(2r-2i+1)(2r-2i+2)}\pi^{2i}\theta^{2r-2i+2}. \tag{25}$$

Combining the first term of the right side of (25) with the summation term and writing

$a_{2r+1}=-\frac{a_{2r-1}}{2r(2r+1)}$ results in

$$\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{(2n+1)^{2(r+1)}}=a_{2r+1}\pi\theta^{2r+1}+\sum_{i=1}^{r+1}a_{2r-2i+2}\pi^{2i}\theta^{2r-2i+2}. \tag{26}$$

This establishes the first part of Theorem 1.1. Following the same technique as in the proof of the above, one can prove the second part of it.

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