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Solutions of General Pentagonal Equation Under Certain Conditions

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Abstract

It was proved by Galois that there are no formulas that can be used to solve "by radicals" general equations of degree 5 or larger. In this paper we solved the general pentagonal equation under certain conditions. We used the "Quartic Formula" and unipodal numbers respectively for condition 1 and conditions 2 to solve the general pentagonal equation. The unipodal case was elaborated with a detailed example.

Key Words: Pentagonal Equations, Unipodal Numbers, Quartic Equation.

1. Introduction

Mathematicians tried over the ages to have formulas to solve polynomial equations. The fruits of their efforts are the Quadratic Formula, and formulas to solve the general cubic equation known as the "Cubic Formula" (Procissi, 1951) and the general quartic equation known as the "Quartic Formula" (Candido, 1941). There are no formulae so far for solving general polynomial equations of degree five or higher. The French mathematician Evariste Galois (Infeld, 1978) proved that there are no formulae for solving general equations of degree 5 or higher. In order to solve the cubic and quartic equations, first of all, the given general equation is transformed into reduced form, and then either Cubic/Quartic Formula or unipodal numbers (Hestenes et al.,1991, Sobczyk, 1995) are used to get the solutions. For pentagonal equation, we used the Quartic Formula and the unipodal numbers.

In section 2, we gave an overview of unipodal numbers. We solved pentagonal equation with an example in section 3 and the references cited were in the section 4.

2. The Unipodal Number System

A unipodal number w in the *standard basis* $\{1, u\}$ has the form $w = w_0 + uw_1$, where $u^2 = 1$ but $u \neq \pm 1$, and w_0, w_1 are complex numbers. The basis $\{u_+, u_-\}$ defined by

$$u_+ = \frac{1}{2}(1 + u) \text{ and } u_- = \frac{1}{2}(1 - u), \qquad (1)$$

and satisfies $u_+ + u_- = 1$ and $u_+ - u_- = u$, is known as *the idempotent basis*, because

$u_+^2 = u_+$ and $u_-^2 = u_-$. One other property of $\{u_+, u_-\}$ is that they mutually annihilating, i.e., $u_+ u_- = 0$. Using the idempotent basis, we can write

$$w = w(u_+ + u_-) = w_+ u_+ + w_- u_-, \text{ where } w_+ = w_0 + w_1 \text{ and } w_- = w_0 - w_1. \qquad (2)$$

We can also recover the coordinates of the standard basis by

$$w_0 = \frac{1}{2}(w_+ + w_-) \qquad \text{and} \qquad w_1 = \frac{1}{2}(w_+ - w_-).$$

(3)

It is noted that the idempotent basis makes calculation simple and the binomial theorem under this basis becomes extremely simple as we see below:

$$(w_+ u_+ + w_- u_-)^n = (w_+)^n u_+^n + (w_-)^n u_-^n = (w_+)^n u_+ + (w_-)^n u_-$$

(4)

The relation defined in (4) is valid for any real *n*. Because of (4) we can extend the definitions of all of the elementary functions in the complex plane to the elementary functions in the unipodal plane. If $f(w)$ is such a function for $w = w_+ u_+ + w_- u_-$, we define

$$f(w) \equiv f(w_+)u_+ + f(w_-)u_-$$

(5)

provided that $f(w_+)$ and $f(w_-)$ are defined. The basic unipodal equation $w^n = r$ can easily be solved using the idempotent basis, with the help of equation (10). Writing $w = w_+ u_+ + w_- u_-$ and $r = r_+ u_+ + r_- u_-$, we have

$$w^n = w_+{}^n u_+ + w_-{}^n u_- = r_+ u_+ + r_- u_-,$$

(6)

So $w_+{}^n = r_+$ and $w_-{}^n = r_-$. It follows that $w_+ = |r_+|^{1/n} \alpha^j$ and $w_+ = |r_-|^{1/a} \alpha^k$ for some integers $0 \le j, k \le n-1$, where α is a primitive *n*th root of unity.

This proves the following theorem.

Theorem 1. For any positive integer n, the unipodal equation $w^n = r$ has n^2 solutions

$$w = \alpha^j r_+{}^{1/n} u_+ + \alpha^k r_-{}^{1/n} u_-$$

for $j, k = 0, 1, \ldots, n-1$, where $\alpha \equiv \exp(2\pi i / n)$.

The number of roots to the equation $w^n = r$ can be reduced by adding constraints. The following corollary follows immediately from the theorem, by noting that $w_+ w_- = \rho \ne 0$, is equivalent to $w_- = \rho / w_+$.

Corollary 1. The unipodal equation $w^n = r$, subject to the constraint $w_+ w_- = \rho$, for a nonzero complex number ρ, has the n solutions

$$w = \alpha^j r_+{}^{1/n} u_+ + \frac{\rho}{\alpha^j r_+{}^{1/n}} u_-$$

$j = 0, 1, \ldots, n-1$, where $\alpha \equiv \exp(2\pi i / n)$, and $r_+{}^{1/n}$ denotes any *n*th root of the complex number r_+.

3. Solutions to Pentagonal Equation

Let us consider the general pentagonal equation

$$ax^5 + 5bx^4 + cx^3 + dx^2 + ex + f = 0. \quad (a \neq 0) \tag{7}$$

By substitution of $x = y - \frac{b}{a}$ in (7), we obtain

$$a\left(y - \frac{b}{a}\right)^5 + 5b\left(y - \frac{b}{a}\right)^4 + c\left(y - \frac{b}{a}\right)^3 + d\left(y - \frac{b}{a}\right)^2 + e\left(y - \frac{b}{a}\right) + f = 0. \tag{8}$$

Expanding and simplifying we obtain the reduced pentagonal equation

$$ay^5 + \left(\frac{-10a^3b^2 + a^4c}{a^4}\right)y^3 + \left(\frac{20a^2b^3 - 3a^3bc}{a^4}\right)y^2$$
$$+\left(\frac{-15ab^4 + 3a^2b^2c - 2a^3bd + a^4e}{a^4}\right)y + \left(\frac{4b^5 + a^2b^2d - ab^3c - a^3be + a^4f}{a^4}\right) = 0.$$

(9)

From equation (9), we observe that if the constant term

$$\left(\frac{4b^5 + a^2b^2d - ab^3c - a^3be + a^4f}{a^4}\right) = 0,$$

(10)

which we call the *first condition*, the reduced pentagonal equation takes the form

$$Ay^5 + By^3 + Cy^2 + Dy = 0,$$

(11)

that can be factored as $y(Ay^4 + By^2 + Cy + D) = 0$, and it provides us with either $y = 0$ or

$$Ay^4 + By^2 + Cy + D = 0,$$

(12)

that is a quartic equation and can be solved by using the Quartic Formula.

If we impose condition that the coefficient of the quadratic term in equation (9) is zero, i.e.

$$\left(\frac{20a^2b^3-3a^3bc}{a^4}\right)=0 \Rightarrow 20b^2-3ac=0$$

(13)

which we call the *second condition*, the equation (9) terns into the form

$$Ay^5+By^3+Cy+D=0,$$

(14)

that is a reduced pentagonal equation of having odd powers of y. We solve the equation of the type in (14) by using the unipodal numbers.

Theorem 2. The reduced pentagonal equation $x^5+5ax^3+bx+c=0$ has the solutions, for $j=0,\ 1,\ 2,\ 3,\ 4,$

$$x=\frac{1}{2}\left(\alpha^j\sqrt[5]{s+t}+\frac{\rho}{\alpha^j\sqrt[5]{s+t}}\right),$$

(15)

where $\alpha=\exp(2\pi i/5)$ is a primitive 5th root of unity, $\rho=-4a,\ b=\frac{5}{16}\rho^2,\ s=-16c,$ and $t=\sqrt{s^2-\rho^5}=16\sqrt{c^2-4a^5}.$

Proof. The unipodal equation $w^5=r$, where $r=s+ut$, is equivalent in the standard basis to $(x+yu)^5=s+tu,$ or $(x^5+10x^3y^2+5xy^4)+(5x^4y+10x^2y^3+y^5)u=s+ut.$ Equating the complex scalar parts we have

$$x^5+10x^3y^2+5xy^4-s=0.$$

(16)

$$\Rightarrow 16x^5-10x^3(x^2-y^2)+5x(y^2-x^2)(y^2-x^2+2x^2)-s=0.$$

(17)

Now substituting $x^2-y^2=\rho$ in (17), we obtain

$$\Rightarrow 16x^5-10\rho x^3+5x(-\rho)(-\rho+2x^2)-s=0.$$

$$\Rightarrow x^5 + 5\left(-\frac{1}{4}\right)\rho x^3 + \left(\frac{5}{16}\right)\rho^2 x - \frac{1}{16}s = 0.$$

(18)

The constraint $w_+ w_- = \rho$ further implies that

$$\rho^5 = (w_+ w_-)^5 = w_+^5 w_-^5 = r_+ r_- = s^2 - t^2,$$

which gives $t = \sqrt{s^2 - \rho^5}$. By letting $\rho = -4a,\ b = \frac{5}{16}\rho^2$ and $s = -16c$, so $t = 16\sqrt{c^2 - 4a^5}$, equation (18) becomes the reduced pentagonal equation $x^5 + 5ax^3 + bx + c = 0.$ Since $r_+ = s + t$, the desired solution (15) is obtained by taking the complex scalar part of the solution given in the Corollary 1, using (3).

Example 1. Find the solution of the reduced pentagonal equation $x^5 - 10x^3 + 20x + 4 = 0.$

Solution. Here $a = -2,\ b = 30,\ c = 4,\ \rho = 8,\ s = -16c = -64,$ and $t = 64\sqrt{7}i.$ Then

$s + t = -64 + 64\sqrt{7}i = 64(-1 + \sqrt{7}) = 2^{15/2}\exp(0.615\pi i)$ and it implies $\sqrt[5]{s+t} = 2^{3/2}\exp(0.123\pi i).$

Using equation (21), we can write

$$x = \frac{1}{2}\left[2^{3/2}\exp(0.123\pi i)\alpha^j + \frac{8}{2^{3/2}\exp(0.123\pi i)\alpha^j}\right]$$

$$= 2^{1/2}\left(\exp(0.123\pi i)\alpha^j + \exp(-0.123\pi i)\alpha^{-j}\right), \text{ for } j = 0, 1, 2, 3, 4.$$

I. For $j = 0,\ x = 2^{1/2}\left[\exp(0.123\pi i) + \exp(-0.123\pi i)\right] = 2^{1/2}\left[2\cos(0.123\pi)\right] = 2.619875175$.

II. For $j = 1$,

$$x = 2^{1/2}\left[\exp(0.123\pi i)\exp(2\pi i/5) + \exp(-0.123\pi i)\exp(-2\pi i/5)\right]$$

$$= 2^{1/2}\left[\exp((0.123 + 2/5)\pi i) + \exp(-(0.123 + 2/5)\pi i\right] = 2^{1/2}\left[2\cos(0.123 + 2/5)\pi\right]$$

$$= -0.204194823 .$$

III. For $j = 2$,

$$x = 2^{1/2}\left[\exp(0.123\pi i)\exp(4\pi i/5) + \exp(-0.123\pi i)\exp(-4\pi i/5)\right]$$

$$= 2^{1/2}\left[\exp((0.123+4/5)\pi i)+\exp(-(0.123+4/5)\pi i)\right] = 2^{1/2}\left[2\cos(0.123+4/5)\pi\right]$$

$$= -2.746074516$$

IV. For $j = 3$,

$$x = 2^{1/2}\left[\exp(0.123\pi i)\exp(6\pi i/5)+\exp(-0.123\pi i)\exp(-6\pi i/5)\right]$$

$$= 2^{1/2}\left[\exp((0.123+6/5)\pi i)+\exp(-(0.123+6/5)\pi i)\right] = 2^{1/2}\left[2\cos(0.123+6/5)\pi\right]$$

$$= -1.4929955855 .$$

V. For $j = 4$,

$$x = 2^{1/2}\left[\exp(0.123\pi i)\exp(8\pi i/5)+\exp(-0.123\pi i)\exp(-8\pi i/5)\right]$$

$$= 2^{1/2}\left[\exp((0.123+8/5)\pi i)+\exp(-(0.123+8/5)\pi i)\right] = 2^{1/2}\left[2\cos(0.123+8/5)\pi\right]$$

$$= 1.823366727 .$$

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