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Λ_r-homeomorphisms and Λ_r^*-homeomorphisms

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Abstract

In this paper, the concepts of Λ_r-homeomorphisms and Λ_r^*-homeomorphisms are introduced and their basic properties are investigated. In particular, it has been shown that Λ_r^*-homeomorphisms form a group under composition.

Key words: Λ_r-open, Λ_r-homeomorphism and Λ_r^*-homeomorphism.
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1. Introduction

The notion of homeomorphisms plays a dominant role in topology and so many authors introduced varies types of homeomorphisms in topological spaces. In 1995, Maki, Devi and Balachandran [2] introduced the concepts of semi-generalized homeomorphisms and generalized semi-homeomorphisms and studied some semi topological properties. Devi and Balachandran [1] introduced a generalization of α-homeomorphism in 2001. Recently, Devi, Vigneshwaran, Vadivel and Vairamanickam [3,6] introduced $g^*\alpha c$-homeomorphisms and rgα-homeomorphisms and obtained some topological properties. The purpose of this paper is to introduce the concepts of homeomorphisms by using Λ_r-open sets. The authors [4] have recently introduced and studied Λ_r-sets, Λ_r-open sets, Λ_r-regular spaces and Λ_r-normal spaces. In this paper, we introduce the concepts Λ_r-homeomorphisms and Λ_r^*-homeomorphisms and investigate their basic properties. The most important property is that the set of all Λ_r^*-homeomorphisms is a group under composition of functions.

Throughout the paper, (X,τ) (or simply X) will always denote a topological space. For a subset S of a topological space X, S is called regular-open [5] if S = *Int cl* S. The complement $S^c = X \setminus S$ of a regular-open set S is called the regular-closed set. The family of all regular-open sets (resp. regular-closed sets) in (X,τ) will be denoted by $RO(X,\tau)$ (resp. $RC(X,\tau)$). A subset S of a topological space (X,τ) is called a Λ_r-set [4] if $S = \Lambda_r(S)$ where $\Lambda_r(S) = \cap\{G : G \in RO(X,\tau) \text{ and } S \subseteq G\}$. The collection of all Λ_r-sets in (X,τ) is denoted by $\Lambda_r(X,\tau)$.

2. Preliminaries

Throughout this paper, we adopt the notations and terminology of [4]. Let A be a subset of a space (X,τ). Then A is called a Λ_r-closed set if $A = S \cap C$ where S is a Λ_r-set and C is a closed set. The complement of a Λ_r-closed set is called Λ_r-open. The collection of all Λ_r-open (resp. Λ_r-closed) sets in (X,τ) is denoted by $\Lambda_rO(X,\tau)$ (resp. $\Lambda_rC(X,\tau)$). We note that every open set is Λ_r-open; arbitrary union of Λ_r-open sets is Λ_r-open and arbitrary intersection of Λ_r-closed sets is Λ_r-closed. A point $x \in X$ is called a Λ_r-cluster point of A if for every Λ_r-open set U containing x, $A \cap U \neq \varnothing$. The set of all Λ_r-cluster points of A is called the Λ_r-closure of A and it is denoted by Λ_r-*cl*(A). Then Λ_r-*cl*(A) is the intersection of Λ_r-closed sets containing A and it is the smallest Λ_r-closed set containing A. Also A is Λ_r-closed if and only if A = Λ_r-

cl(A). The union of Λ_r-open sets contained in A is called Λ_r-interior of A and it is denoted by Λ_r-*int*(A).

Definition 2.1

A function f : X → Y is called

(a) Λ_r-continuous if $f^{-1}(V)$ is a Λ_r-open set in X for each open set V in Y.

(b) Λ_r-irresolute if $f^{-1}(V)$ is a Λ_r-open set in X for each Λ_r-open set V in Y.

(c) Λ_r-open if the image of each open set in X is a Λ_r-open set in Y.

(d) Λ_r-closed if the image of each closed set in X is a Λ_r-closed set in Y.

Lemma 2.2

Let f : X → Y be a function where X and Y are topological spaces. Then f is Λ_r-continuous if and only if the inverse image of each closed set in Y is Λ_r-closed in X.

Lemma 2.3

A function f : X → Y is Λ_r-irresolute if and only if $f^{-1}(V)$ is a Λ_r-closed set in X for every Λ_r-closed set V in Y.

3. Λ_r-homeomorphism and Λ_r^*-homeomorphism

In this section, we introduce the concepts of Λ_r-homeomorphisms and Λ_r^*-homeomorphisms in topological spaces and we investigate the group structure of the set of all Λ_r^*-homeomorphisms.

Definition 3.1

A bijection f : $(X,\tau) \to (Y,\sigma)$ is called Λ_r-homeomorphism if both f and f^{-1} are Λ_r-continuous.

We denote the family of all Λ_r-homeomorphisms of a topological space (X,τ) onto itself by $\Lambda_r H(X,\tau)$.

Theorem 3.2

Every homeomorphism is a Λ_r-homeomorphism.

Proof

Let f : $(X,\tau) \to (Y,\sigma)$ be a homeomorphism. Then f is bijective and both f and f^{-1} are continuous. Since every continuous function is Λ_r-continuous, f and f^{-1} are Λ_r-continuous. This shows that f is a Λ_r-homeomorphism.

Remark 3.3

The converse of the above theorem need not be true, as shown in the following example.

Example 3.4

Let X = Y = {a,b,c}, τ = {X,Ø,{a},{b},{a,b},{b,c}} and σ = {Y,Ø,{b},{c},{b,c}}.

Then Λ_rO(X,τ) = τ and Λ_rO(Y,σ) = {Y, Ø,{b},{c},{b,c},{a,c},{a,b}}.

Let f : (X,τ) → (Y,σ) be defined by f (a) = c, f (b) = b and f (c) = a.

Then f is Λ_r-homeomorphism. Since f ({b, c}) = {a, b} is not open in (Y,σ), f $^{-1}$ is not continuous that implies f is not a homeomorphism.

Theorem 3.5

Let f : (X,τ) → (Y,σ) be a bijective Λ_r-continuous function. Then the following are equivalent:

(a) f is Λ_r-open

(b) f is Λ_r-homeomorphism

(c) f is Λ_r-closed

Proof

Suppose (a) holds. Let V be open in (X,τ). Then by (a), f (V) is Λ_r-open in (Y,σ). But f (V) = (f $^{-1}$)$^{-1}$(V) and so (f $^{-1}$)$^{-1}$(V) is Λ_r-open in (Y,σ). This shows that f $^{-1}$ is Λ_r-continuous and it proves (b).

Suppose (b) holds. Let F be a closed set in (X,τ). By (b), f $^{-1}$ is Λ_r-continuous and so (f $^{-1}$)$^{-1}$(F) = f (F) is Λ_r-closed in (Y,σ). This proves (c).

Suppose (c) holds. Let V be open in (X,τ). Then V^c is closed in (X,τ). By (c), f (V^c) is Λ_r-closed in (Y,σ). But f (V^c) = (f (V))^c. This implies that (f (V))^c is Λ_r-closed in (Y,σ) and so f (V) is Λ_r-open in (Y,σ). This proves (a).

Remark 3.6

The composition of two Λ_r-homeomorphisms need not be Λ_r-homeomorphism, as shown in the following example.

Example 3.7

Let X = Y = Z = {a,b,c}, τ = {X,Ø,{a},{b},{a,b},{b,c}}, σ = {Y,Ø,{b},{c},{b,c}} and γ = {Z,Ø,{a},{b},{a,b},{b,c}}. Then Λ_rO(X,τ) = τ, Λ_rO(Y,σ) = {Y,Ø,{b},{c},{b,c}, {a,c},{a,b}} and Λ_rO(Z,γ) = γ. Define f : (X,τ) → (Y,σ) by f (a) = c, f (b) = b and f (c) = a and define g : (Y,σ) → (Z,γ) by g(a) = c, g(b) = a and g(c) = b. Then f and g are Λ_r-homeomorphisms. Here g ∘ f is not Λ_r-continuous since {b,c} is open in (Z,γ) but (g ∘ f) $^{-1}$({b,c}) = {a,c} is not Λ_r-open in (X,τ) and so g ∘ f is not Λ_r-homeomorphism.

Definition 3.8

A bijection f : $(X,\tau) \rightarrow (Y,\sigma)$ is said to be Λ_r^*-homeomorphism if both f and f $^{-1}$ are Λ_r-irresolute.

We say that spaces (X,τ) and (Y,σ) are Λ_r^*-homeomorphic if there exists a Λ_r^*-homeomorphism from (X,τ) onto (Y,σ). We denote the family of all Λ_r^*-homeomorphisms of a topological space (X,τ) onto itself by $\Lambda_r^*H(X,\tau)$.

Theorem 3.9

Every Λ_r^*-homeomorphism is a Λ_r-homeomorphism.

Proof

Let f : $(X,\tau) \rightarrow (Y,\sigma)$ be a Λ_r^*-homeomorphism. Then f is bijective, Λ_r-irresolute and f $^{-1}$ is Λ_r-irresolute. Since every Λ_r-irresolute function is Λ_r-continuous, f and f $^{-1}$ are Λ_r-continuous and so f is a Λ_r-homeomorphism.

Remark 3.10

The following example shows that the converse of the above theorem need not be true.

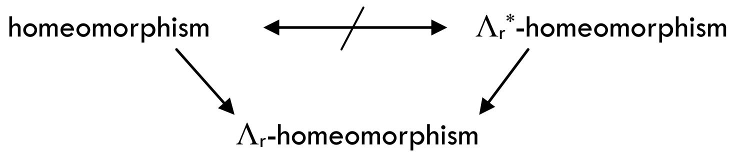
Example 3.11

Let X, Y, τ, σ and f be defined as in Example 3.4. Then f is a Λ_r-homeomorphism but not a Λ_r^*-homeomorphism since {a, c} is Λ_r-open in (Y,σ) but f $^{-1}$({a, c}) = {a, c} is not Λ_r-open in (X,τ) and so f is not Λ_r-irresolute.

Examples can be constructed to show that the concepts of homeomorphisms and Λ_r^*-homeomorphism are independent.

Remark 3.12

From the above discussions, we have the following implications.



where "A ⟶ B" means A implies B but not conversely and

"A ⟷̸ B" means A and B are independent of each other.

Theorem 3.13

If f : $(X,\tau) \rightarrow (Y,\sigma)$ is a Λ_r^*-homeomorphism, then Λ_r-*cl*(f $^{-1}$(B)) = f $^{-1}$(Λ_r-*cl*(B)) for every B $\subseteq$ Y.

Proof

Let f : $(X,\tau) \rightarrow (Y,\sigma)$ be a Λ_r^*-homeomorphism. Then by Definition 3.8, both f and f $^{-1}$ are Λ_r-irresolute and f is bijective. Let B $\subseteq$ Y. Since Λ_r-*cl*(B) is a Λ_r-closed set in (Y,σ), using

Lemma 2.3, $f^{-1}(\Lambda_r\text{-}cl(B))$ is Λ_r-closed in (X,τ). But $\Lambda_r\text{-}cl(f^{-1}(B))$ is the smallest Λ_r-closed set containing $f^{-1}(B)$.

Therefore $\Lambda_r\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(\Lambda_r\text{-}cl(B))$. $\rightarrow$ (1)

Again, $\Lambda_r\text{-}cl(f^{-1}(B))$ is Λ_r-closed in (X,τ). Since f^{-1} is Λ_r-irresolute, $f(\Lambda_r\text{-}cl(f^{-1}(B)))$ is Λ_r-closed in (Y,σ). Now, $B = f(f^{-1}(B)) \subseteq f(\Lambda_r\text{-}cl(f^{-1}(B)))$. Since $f(\Lambda_r\text{-}cl(f^{-1}(B)))$ is Λ_r-closed and $\Lambda_r\text{-}cl(B)$ is the smallest Λ_r-closed set containing B, $\Lambda_r\text{-}cl(B) \subseteq f(\Lambda_r\text{-}cl(f^{-1}(B)))$ that implies $f^{-1}(\Lambda_r\text{-}cl(B)) \subseteq f^{-1}(f(\Lambda_r\text{-}cl(f^{-1}(B)))) = \Lambda_r\text{-}cl(f^{-1}(B))$.

That is, $f^{-1}(\Lambda_r\text{-}cl(B)) \subseteq \Lambda_r\text{-}cl(f^{-1}(B))$ $\rightarrow$ (2)

From (1) and (2), $\Lambda_r\text{-}cl(f^{-1}(B)) = f^{-1}(\Lambda_r\text{-}cl(B))$.

Corollary 3.14

If $f : (X,\tau) \rightarrow (Y,\sigma)$ is a Λ_r^*-homeomorphism, then $\Lambda_r\text{-}cl(f(B)) = f(\Lambda_r\text{-}cl(B))$ for every $B \subseteq X$.

Proof

Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be a Λ_r^*-homeomorphism. Since f is Λ_r^*-homeomorphism, f^{-1} is also a Λ_r^*-homeomorphism. Therefore by Theorem 3.13, it follows that $\Lambda_r\text{-}cl(f(B)) = f(\Lambda_r\text{-}cl(B))$ for every $B \subseteq X$.

Corollary 3.15

If $f : (X,\tau) \rightarrow (Y,\sigma)$ is a Λ_r^*-homeomorphism, then $f(\Lambda_r\text{-}int(B)) = \Lambda_r\text{-}int(f(B))$ for every $B \subseteq X$.

Proof

Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be a Λ_r^*-homeomorphism. For any set $B \subseteq X$, $\Lambda_r\text{-}int(B) = (\Lambda_r\text{-}cl(B^c))^c$.

$f(\Lambda_r\text{-}int(B)) = f((\Lambda_r\text{-}cl(B^c))^c) = (f(\Lambda_r\text{-}cl(B^c)))^c$. Then using Corollary 3.14, we see that

$f(\Lambda_r\text{-}int(B)) = (\Lambda_r\text{-}cl(f(B^c)))^c = \Lambda_r\text{-}int(f(B))$.

Corollary 3.16

If $f : (X,\tau) \rightarrow (Y,\sigma)$ is a Λ_r^*-homeomorphism, then for every $B \subseteq Y$,

$f^{-1}(\Lambda_r\text{-}int(B)) = \Lambda_r\text{-}int(f^{-1}(B))$.

Proof

Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be a Λ_r^*-homeomorphism. Since f is Λ_r^*-homeomorphism, f^{-1} is also a Λ_r^*-homeomorphism. Therefore by Corollary 3.15, $f^{-1}(\Lambda_r\text{-}int(B)) = \Lambda_r\text{-}int(f^{-1}(B))$ for every $B \subseteq Y$.

Theorem 3.17

If $f : (X,\tau) \rightarrow (Y,\sigma)$ and $g : (Y,\sigma) \rightarrow (Z,\gamma)$ are Λ_r^*-homeomorphisms, then the composition $g \circ f : (X,\tau) \rightarrow (Z,\gamma)$ is also Λ_r^*-homeomorphism.

Proof

Let U be a Λ_r-open set in (Z,γ). Since g is Λ_r^*-homeomorphism, g is Λ_r-irresolute and so $g^{-1}(U)$ is Λ_r-open in (Y,σ). Since f is Λ_r^*-homeomorphism, f is Λ_r-irresolute and so $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is Λ_r-open in (X,τ). This implies that $g \circ f$ is Λ_r-irresolute.

Again, let G be Λ_r-open in (X,τ). Since f is Λ_r^*-homeomorphism, f^{-1} is Λ_r-irresolute and so $(f^{-1})^{-1}(G) = f(G)$ is Λ_r-open in (Y,σ). Since g is Λ_r^*-homeomorphism, g^{-1} is Λ_r-irresolute and so $(g^{-1})^{-1}(f(G)) = g(f(G)) = (g \circ f)(G) = ((g \circ f)^{-1})^{-1}(G)$ is Λ_r-open in (Z,γ). This implies that $(g \circ f)^{-1}$ is Λ_r-irresolute. Since f and g are Λ_r^*-homeomorphism, f and g are bijective and so $g \circ f$ is bijective. This completes the proof.

Theorem 3.18

The set $\Lambda_r^*H(X,\tau)$ is a group under composition of functions.

Proof

Let f, g $\in \Lambda_r^*H(X,\tau)$. Then $f \circ g \in \Lambda_r^*H(X,\tau)$ by Theorem 3.17. Since f is bijective, $f^{-1} \in \Lambda_r^*H(X,\tau)$. This completes the proof.

Theorem 3.19

If $f : (X,\tau) \rightarrow (Y,\sigma)$ is a Λ_r^*-homeomorphism, then f induces an isomorphism from the group $\Lambda_r^*H(X,\tau)$ onto the group $\Lambda_r^*H(Y,\sigma)$.

Proof

Let $f \in \Lambda_r^*H(X,\tau)$. Then define a map $\psi_f : \Lambda_r^*H(X,\tau) \rightarrow \Lambda_r^*H(Y,\sigma)$ by $\psi_f(h) = f \circ h \circ f^{-1}$ for every $h \in \Lambda_r^*H(X,\tau)$. Let $h_1, h_2 \in \Lambda_r^*H(X,\tau)$.

Then $\psi_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1}$

$$= f \circ (h_1 \circ f^{-1} \circ f \circ h_2) \circ f^{-1}$$

$$= (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1})$$

$$= \psi_f(h_1) \circ \psi_f(h_2).$$

Since $\psi_f(f^{-1} \circ h \circ f) = h$, ψ_f is onto. Now, $\psi_f(h) = I$ implies $f \circ h \circ f^{-1} = I$. That implies $h = I$. This proves that ψ_f is one-one. This shows that ψ_f is a isomorphism.

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