On $\Lambda^\delta$-Homeomorphisms In Topological Spaces

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Abstract

In this paper, we first introduce a new class of closed map called $\Lambda^\delta$-closed map. Moreover, we introduce a new class of homeomorphism called $\Lambda^\delta$-Homeomorphism, which are weaker than homeomorphism. We also introduce $\Lambda^\delta^*$-Homeomorphisms and prove that the set of all $\Lambda^\delta$-Homeomorphisms form a group under the operation of composition of maps.

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1. Introduction

The notion homeomorphism plays a very important role in topology. By definition, a homeomorphism between two topological spaces $X$ and $Y$ is a bijective map $f : X \rightarrow Y$ when both $f$ and $f^{-1}$ are continuous. Maki. et al. [5] introduced g-homeomorphisms and gc-homeomorphisms in topological spaces.

In this paper, we first introduce $\Lambda^2$ - closed maps in topological spaces and then we introduce and study $\Lambda^2$-homeomorphisms, which are weaker than homeomorphisms. We also introduce $\Lambda^*$-homeomorphisms. It turns out that the set of all $\Lambda^*$-homeomorphisms forms a group under the operation composition of functions.

2. Preliminaries

Throughout this paper $(X, \tau)$ and $(Y, \sigma)$ represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $(X, \tau)$, $\text{Cl} (A)$, $\text{Int} (A)$ and $X - A$ denote the closure of $A$, the interior of $A$ and the complement of $A$ in $X$, respectively.

We recall the following definitions and some results, which are used in the sequel.

**Definition 2.1:** A subset $A$ of a space $(X, \tau)$ is called: -

1. $\lambda$ - closed [1] if $A = B \cap C$, where $B$ is a $\Lambda$ - set and $C$ is a closed set.
   
   The complement of $\lambda$-closed set is called a $\lambda$ - open set.

2. $\Lambda$ - g-closed [2] if $\text{Cl}_\lambda (A) \subseteq U$, wherever $A \subseteq U$, $U$ is $\lambda$ -open, where $\text{Cl}_\lambda (A)$
   
   [4] is called the $\lambda$ - Closure of $A$. The complement of $\Lambda$ - g-closed set is called a

   $\Lambda$ - g-closed – open set.

   The family of all $\lambda$ - open subsets of a space $(X, \tau)$ shall be denoted by

   $\lambda O (X, \tau)$.

**Definition 2.2:**

A subset $A$ of a space $(X, \tau)$ is called a generalized closed (briefly g-closed) set [6] if $\text{Cl} (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

**Definition 2.5:** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called:

1. $\lambda$ - continuous [1] if $f^{-1} (V)$ is $\lambda$ -open in $(X, \tau)$, for every open set $V$ in $(Y, \sigma)$.
(2) $\lambda$ -irresolute [3] if $f^{-1}(V)$ is $\lambda$ -open in $(X, \tau)$, for every $\lambda$ -open set $V$ in $(Y, \sigma)$.

(3) g-continuous [6] if $f^{-1}(V)$ is g-closed in $(X, \tau)$ for every g-closed set $V$ in $(Y, \sigma)$.

(4) gc-irresolute [6] if $f^{-1}(V)$ is g-closed in $(X, \tau)$ for every g-closed set $V$ in $(Y, \sigma)$.

(5) $\lambda$ -closed [3] if $f(V)$ is $\lambda$ -closed in $(Y, \sigma)$, for every closed set $V$ in $(X, \tau)$.

**Definition 2.6:** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called g-open [6] if $f(V)$ is g-open in $(Y, \sigma)$, for every g-open set $V$ in $(X, \tau)$.

**Definition 2.7:** A bijective function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a:

1) generalized homeomorphism (briefly g-homeomorphism) [7] if $f$ is both g-continuous and g-open.

2) gc-homeomorphism [7] if both $f$ and $f^{-1}$ are gc-irresolute maps.

3. $\Lambda^i$-Closed Sets

**Definition 3.1:** A subset $A$ of a topological space $(X, \tau)$ is called :

1) $\Lambda^i$ - set if $A = A^{\Lambda^i}$, where $A^{\Lambda^i} = \bigcap \{B : B \supseteq A, B \in \lambda\sigma (X, \tau)\}$

2) $\Lambda^i$ - closed set if $A = L \cap F$, where $L$ is $\Lambda^i$ - set and $F$ is $\lambda$ - closed.

The complement of $\Lambda^i$ - set and $\Lambda^i$ - closed set is $V^\lambda_i$ - set and $\Lambda^i$ - open set respectively.

**Proposition 3.2:**

1) [3]Every closed (resp.open) set is $\lambda$-closed (resp. $\lambda$-open) set.

2) Every $\lambda$-closed (resp. $\lambda$-open) set is $\Lambda^i$-closed (resp. $\Lambda^i$-open) set.

**Definition 3.3:** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called :

1) $\Lambda$ -g-continuous if $f^{-1}(V)$ is $\Lambda$ -g-open in $(X, \tau)$, for every open set $V$ in $(Y, \sigma)$.

2) $\Lambda^i$ -continuous if $f^{-1}(V)$ is $\Lambda^i$ -open in $(X, \tau)$, for every open set $V$ in $(Y, \sigma)$.

3) $\Lambda^i$ - $\lambda$ -continuous if $f^{-1}(V)$ is $\Lambda^i$ -open in $(X, \tau)$, for every $\lambda$ -open set $V$ in $(Y, \sigma)$.

4) $\Lambda^i$ - irresolute if $f^{-1}(V)$ is $\Lambda^i$ -open in $(X, \tau)$, for every $\Lambda^i$ -open set $V$ in $(Y, \sigma)$.

5) $\Lambda^i$ - open if $f(V)$ is $\Lambda^i$ -open in $(Y, \sigma)$, for every $\Lambda^i$ -open set $V$ in $(X, \tau)$.

**Definition 3.4:** Let $(X, \tau)$ be a space and $A \subseteq X$. A point $x \in X$ is called

$\Lambda^i$ -cluster point of $A$ if for every $\Lambda^i$ -open set $U$ of $X$ containing $x$, $A \cap U \neq \phi$. The set of all $\Lambda^i$ -cluster points is called the $\Lambda^i$ - closure of $A$ and is denoted by $Cl^{\Lambda^i}(A)$.  

Theorem 3.5: If \( f : (X, \tau) \to (Y, \sigma) \) is \( \Lambda^i \)-irresolute then the map \( f \) is \( \Lambda^i \)-continuous.

**Proof:** Let \( f \) be \( \Lambda^i \)-irresolute. Let \( V \) be an open set in \((Y, \sigma)\). By Proposition 3.2, \( V \) is \( \Lambda^i \)-open in \((Y, \sigma)\). Since \( f \) is \( \Lambda^i \)-irresolute, \( f^{-1}(V) \) is \( \Lambda^i \)-open in \((X, \tau)\). Hence \( f \) is \( \Lambda^i \)-continuous.

Proposition 3.6: Let \( A \) and \( B \) be a subset of a topological space \((X, \tau)\). The following properties hold:

1. \( A \subset Cl^{\Lambda^i}(A) \subset Cl^\Lambda(A) \)
2. \( Cl^{\Lambda^i}(A) = \bigcap \{ F \in \Lambda^i C(X, \tau) / A \subset F \} \)
3. If \( A \subset B \) then \( Cl^{\Lambda^i}(A) \subset Cl^{\Lambda^i}(B) \)
4. \( A \) is \( \Lambda^i \)-closed if and only if \( A = Cl^{\Lambda^i}(A) \).
5. \( Cl^{\Lambda^i}(A) \) is \( \Lambda^i \)-closed.

**Proof:**

1. Let \( x \notin Cl^{\Lambda^i}(A) \). Then \( x \) is not a \( \Lambda^i \)-cluster point of \( A \). So there exists a \( \Lambda^i \)-open set \( U \) containing \( x \) such that \( A \cap U = \emptyset \) and hence \( x \notin A \).

Then \( Cl^{\Lambda^i}(A) \subset Cl^\Lambda(A) \) follows from Proposition 3.2.

2. Suppose \( x \in \bigcap \{ F / A \subset F \text{ and } F \text{ is } \Lambda^\lambda \text{-closed} \} \). Let \( U \) be a \( \Lambda^i \)-open set containing \( x \) such that \( A \cap U = \emptyset \). And so \( A \subset X - U \). But \( X - U \) is \( \Lambda^i \)-closed and hence \( Cl^{\Lambda^i}(A) \subset X - U \). Since \( x \notin X - U \), we obtain \( x \notin Cl^{\Lambda^i}(A) \) which is contrary to the hypothesis. Hence \( Cl^{\Lambda^i}(A) \subset \bigcap \{ F / A \subset F \text{ and } F \text{ is } \Lambda^\lambda \text{-closed} \} \).

Suppose that \( x \in Cl^{\Lambda^i}(A) \), i.e., that every \( \Lambda^i \)-open set of \( X \) containing \( x \) meets \( A \). If \( x \notin \bigcap \{ F / A \subset F \text{ and } F \text{ is } \Lambda^\lambda \text{-closed} \} \), then there exists a \( \Lambda^i \)-closed set \( F \) of \( X \) such that \( A \subset F \) and \( x \notin F \). Therefore \( x \in X - F \in \Lambda^i O(X, \tau) \). Hence \( X - F \) is a \( \Lambda^i \)-open set of \( X \) containing \( x \), but \( (X - F) \cap A = \emptyset \). But this is a contradiction. Hence \( Cl^{\Lambda^i}(A) \subset \bigcap \{ F / A \subset F \text{ and } F \text{ is } \Lambda^\lambda \text{-closed} \} \). Thus,
$Cl^\xi(A) = \cap\{ F \mid A \subset F \text{ and } F \text{ is } \Lambda^\xi - closed \}$.

(3) Let $x \notin Cl^\xi(B)$. Then there exists a $\Lambda^\xi$-open set $V$ containing $x$ such that $B \cap V = \phi$. Since $A \subset B, A \cap V = \phi$ and hence $x$ is not a $\Lambda^\xi$-cluster point of $A$. Therefore $x \notin Cl^\xi(A)$.

(4) Let $A$ is $\Lambda^\xi$ closed. Let $x \notin A$. Then $x$ belongs to the $\Lambda^\xi$-open $X-A$. Then a $\Lambda^\xi$-open set $X-A$ containing $x$ and $A \cap (X - A) = \phi$. Hence $x \notin Cl^\xi(A)$. By (1), we get $A = Cl^\xi(A)$. Conversely, Suppose $A = Cl^\xi(A)$. By (2),

$A = \cap \{ F \in \Lambda \cap C(X, \tau) / A \subset F \}$. Hence $A$ is $\Lambda^\xi$-closed.

(5) By (1) and (3), we have $Cl^\xi(A) \subset Cl^\xi(Cl^\xi(A))$. Let

$x \in Cl^\xi(Cl^\xi(A))$. Hence $x$ is a $\Lambda^\xi$-cluster point of $Cl^\xi(A)$. That implies for every $\Lambda^\xi$-open set $U$ containing $x, Cl^\xi(A) \cap U \neq \phi$. Let

$p \in Cl^\xi(A) \cap U$. Then for every $\Lambda^\xi$-open set $G$ containing $p, A \cap G \neq \phi$, since

$p \in Cl^\xi(A)$. Since $U$ is $\Lambda^\xi$-open and $x, p \in U, A \cap U \neq \phi$. Hence

$x \in Cl^\xi(A)$. Hence $Cl^\xi(A) = Cl^\xi(Cl^\xi(A))$. By (4), $Cl^\xi(A)$ is $\Lambda^\xi$-closed.

**Definition 3.7:** A subset $A$ of a topological space $(X, \tau)$ is said to be $\xi$-locally closed if $A = S \cap P$, where $S$ is $\xi$-open in $X$ and $P$ is $\xi$-closed in $X$.

**Lemma 3.8:** Let $A$ be $\xi$-closed subset of a topological space $(X, \tau)$. Then we have,

1) $A = T \cap Cl^\xi(A)$, where $T$ is a $\Lambda^\xi$-set.

2) $A = A^\xi \cap Cl^\xi(A)$

**Lemma 3.9:** A subset $A \subseteq (X, \tau)$ is $\xi - g - closed$ iff $Cl^\xi(A) \subseteq A^\xi$.

**Proposition 3.10:** For a subset $A$ of a topological space the following conditions are equivalent.

(1) $A$ is $\xi$ - closed.

(2) $A$ is $\Lambda - g$-closed and $\xi$-locally closed

(3) $A$ is $\Lambda - g$-closed and $\Lambda^\xi$ - closed.

**Proof:** (1) $\Rightarrow$ (2) Every $\xi$-closed set is both $\Lambda$-g-closed and $\xi$-locally closed.

(2) $\Rightarrow$ (3) This is obvious from the fact that every $\xi$-locally closed is a $\Lambda^\xi$-closed.
(3) $\Rightarrow$ (1) $A$ is $\Lambda$-g-closed, so by Lemma 3.9, $\text{Cl}^\lambda (A) \subseteq A^{\lambda^*}$. $A$ is $\Lambda^*$-closed, so by Lemma 3.8, $A = A^{\lambda^*} \cap \text{Cl}^\lambda (A)$. Hence $A = \text{Cl}^\lambda (A)$, i.e., $A$ is $\lambda^*$-closed.

4. $\Lambda^*$-closed Maps

In this section, we introduce $\Lambda^*$-closed maps. $\Lambda^*$-open maps, $\Lambda^*$-closed maps, $\Lambda^*$-open maps, $\Lambda^*$-closed maps and $\Lambda^*$-open maps.

Definition 4.1:

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be a $\Lambda^*$-closed map if the image of every closed set in $(X, \tau)$ is $\Lambda^*$-set in $(Y, \sigma)$.

Example 4.2:

(a) Let $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b, c\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}\}$. Define a map $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f (a) = c, f (b) = d, f (c) = b$ and $f (d) = a$.

Then $f$ is a $\Lambda^*$-closed.

(b) Let $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{b\}, \{b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map $f (\{c, d\}) = \{c, d\}$ is not a $\Lambda^*$-set. Hence $f$ is not a $\Lambda^*$-closed map.

Definition 4.3: A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\Lambda^*$-closed if the image of every closed set in $(X, \tau)$ is $\Lambda^*$-closed in $(Y, \sigma)$.

Example 4.4:

(a) Let $X = Y = \{a, b, c, d, e\}$, $\tau = \{\phi, X, \{a\}, \{d, e\}, \{a, d, e\}, \{b, c, d\}, \{a, b, c, d\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b, c, d\}\}$. Define a map $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f (a) = d, f (b) = e, f (c) = a, f (d) = c$ and $f (e) = b$. Then $f$ is a $\Lambda^*$-closed map.

(b) Let $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, b\}, \{a, b, c\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b, c\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity map $f (\{c, d\}) = \{c, d\}$ is not $\Lambda^*$-closed set. Hence $f$ is not a $\Lambda^*$-closed map.

Definition 4.5: A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\Lambda$--g-closed if the image of every closed set in $(X, \tau)$ is $\Lambda$--g-closed in $(Y, \sigma)$.  

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Example 4.6:
(i) Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$ and
$\sigma = \{\emptyset, Y, \{a, b\}, \{c, d\}, \{a, c, d\}\}$. Define an identity map $f : X \to Y$ is $\Lambda$-g-closed.

(ii) Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and
$\sigma = \{\emptyset, Y, \{a, b\}, \{c, d\}, \{a, c, d\}\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by
$f(a) = d$, $f(b) = b$, $f(c) = a$, $f(d) = c$. But $\{b, c\}$ is not a $\Lambda$-g-closed. Hence $f$ is not $\Lambda$-g-closed.

Definition 4.7:
A map $f : (X, \tau) \to (Y, \sigma)$ is said to be $\Lambda^\lambda - \Lambda$-closed if the image of every $\lambda$-closed set in $(X, \tau)$ is $\Lambda^\lambda$-closed in $(Y, \sigma)$.

Example 4.8:
The function $f$ which is defined in example 4.2(a) is $\Lambda^\lambda - \Lambda$-closed.

Proposition 4.9:
Every $\Lambda^\lambda - \Lambda$-closed map is a $\Lambda^\lambda$-closed map.

Proof:
Let $f : (X, \tau) \to (Y, \sigma)$ be a $\Lambda^\lambda - \Lambda$-closed map. Let $B$ be a closed set in $(X, \tau)$ and hence $B$ is a $\lambda$-closed set in $(X, \tau)$. By assumption $f(B)$ is $\Lambda^\lambda$-closed in $(Y, \sigma)$. Hence $f$ is a $\Lambda^\lambda$-closed map.

Theorem 4.10:
Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two mappings such that their composition $gof : (X, \tau) \to (Z, \eta)$ be a $\Lambda^\lambda$-closed mapping. Then the following statements are true if $f$ is continuous and surjective then $g$ is $\Lambda^\lambda$-closed.

Proof:
1) Let $A$ be a closed set in $(Y, \sigma)$. Since $f$ is continuous, $f^{-1}(A)$ is closed in $(X, \tau)$ and since $gof$ is $\Lambda^\lambda$-closed, $(gof)(f^{-1}(A))$ is $\Lambda^\lambda$-closed in $(Z, \eta)$. i.e., $g(A)$ is $\Lambda^\lambda$-closed in $(Z, \eta)$.
2) Let $B$ be a closed set of $(X, \tau)$. Since $gof$ is $\Lambda^I$-closed, $(gof)(B)$ is $\Lambda^I$-closed in $(Z, \eta)$. Since $g$ is $\Lambda^I$-irresolute $g^{-1}((gof)(B))$ is $\Lambda^I$-closed in $(Y, \sigma)$, i.e., $f$ is $\Lambda^I$-closed in $(Y, \sigma)$. Since $g$ is injective. Thus $f$ is a $\Lambda^I$-closed map.

3) Let $A$ be a closed set of $(Y, \sigma)$. Since $f$ is $g$-continuous, $f^{-1}(A)$ is $g$-closed in $(X, \tau)$. Since $(X, \tau)$ is a $T_{1/2}$-space, $f^{-1}(A)$ is closed in $(X, \tau)$ and so as in (i), $g$ is a $\Lambda^I$-closed map.

**Theorem 4.11:** Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two mappings such that their composition $gof : (X, \tau) \to (Z, \eta)$ be a $\Lambda^I$-$\lambda$-closed mapping. Then the following is true if

1) $f$ is $\lambda$-irresolute and surjective then $g$ is $\Lambda^I$-$\lambda$-closed.

2) $g$ is $\Lambda^I$-irresolute and injective then $f$ is $\Lambda^I$-$\lambda$-closed.

**Proof:**

1) Let $A$ be a $\lambda$-closed set of $(Y, \sigma)$. Since $f$ is $\lambda$-irresolute, $f^{-1}(A)$ is $\lambda$-closed in $(X, \tau)$ and since $gof$ is $\Lambda^I$-$\lambda$-closed, $(gof)(f^{-1}(A))$ is $\Lambda^I$-closed in $(Z, \eta)$.

i.e., $g(A)$ is $\Lambda^I$-closed in $(Z, \eta)$. Hence $g$ is $\Lambda^I$-$\lambda$-closed.

2) Let $B$ be a $\lambda$-closed set of $(X, \tau)$. Since $gof$ is $\Lambda^I$-$\lambda$-closed, $(gof)(B)$ is $\Lambda^I$-closed in $(Z, \eta)$. Since $g$ is $\Lambda^I$-irresolute $g^{-1}((gof)(B))$ is $\Lambda^I$-closed in $(Y, \sigma)$, i.e., $f$ is $\Lambda^I$-$\lambda$-closed in $(Y, \sigma)$, since $g$ is injective. Thus $f$ is $\Lambda^I$-$\lambda$-closed map.

**Definition 4.12:**

A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a $\Lambda^I$-open map if the image $f(A)$ is $\Lambda^I$-open in $(Y, \sigma)$ for each open set $A$ in $(X, \tau)$.

**Proposition 4.13:**

For any bijection $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent.

1) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\Lambda^I$-continuous.

2) $f$ is a $\Lambda^I$-open map and

3) $f$ is a $\Lambda^I$-closed map.

**Proof:**

(1) $\Rightarrow$ (2): Let $U$ be an open set of $(X, \tau)$. By assumption $(f^{-1})^{-1}(U) = f(U)$ is $\Lambda^I$-open in $(Y, \sigma)$ and so $f$ is $\Lambda^I$-open.
(2) \( \Rightarrow \) (3): Let \( F \) be a closed set of \( (X, \tau) \). Then \( X-F \) is open in \( (X, \tau) \). By assumption, \( f \) (\( X-F \)) is \( \Lambda^i \)-open in \( (Y, \sigma) \) and therefore
\[
f (X-F) = (Y - f (F)) \text{ is } \Lambda^i \text{-open in } (Y, \sigma) \text{ and therefore } f (F) \text{ is} \Lambda^i \text{-closed in } (Y, \sigma).
\]
Hence \( f \) is \( \Lambda^i \)-closed.

(3) \( \Rightarrow \) (1): Let \( F \) be a closed set of \( (X, \tau) \). By assumption, \( f (F) \) is \( \Lambda^i \)-closed in \( (Y, \sigma) \). But \( f (F) = (f^{-1})^{-1}(F) \) and therefore \( f^{-1} \) is \( \Lambda^i \)-continuous on \( Y \).

**Definition 4.14:** A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be a \( \Lambda^i - \lambda \)-open map if the image \( f (A) \) is \( \Lambda^i \)-open in \( (Y, \sigma) \) for each \( \lambda \)-open set \( A \) in \( (X, \tau) \).

**Proposition 4.15:**

For any bijection \( f : (X, \tau) \rightarrow (Y, \sigma) \), the following statements are equivalent.

1) \( f^{-1} : (Y, \sigma) \rightarrow (X, \tau) \) is \( \Lambda^i - \lambda \)-continuous.

2) \( f \) is a \( \Lambda^i - \lambda \)-open map and

3) \( f \) is a \( \Lambda^i - \lambda \)-closed map

**Proof:**

(1) \( \Rightarrow \) (2) Let \( U \) be an \( \lambda \)-open set of \( (X, \tau) \). By assumption \( (f^{-1})^{-1}(U) = f^{-1}(U) \) is \( \Lambda^i \)-open in \( (Y, \sigma) \) and so \( f \) is \( \Lambda^i - \lambda \)-open.

(2) \( \Rightarrow \) (3) Let \( F \) be a \( \lambda \)-closed set of \( (X, \tau) \). Then \( X-F \) is \( \lambda \)-open in \( (X, \tau) \). By assumption, \( f \) (\( X-F \)) is \( \Lambda^i \)-open in \( (Y, \sigma) \). i.e., \( f \) (\( X-F \)) = \( Y - f \) (\( F \)) is \( \Lambda^i \)-open in \( (Y, \sigma) \) and there \( f \) (\( F \)) is \( \Lambda^i \)-closed in \( (Y, \sigma) \). Hence \( f \) is \( \Lambda^i \)-closed.

(3) \( \Rightarrow \) (1) Let \( F \) be a \( \lambda \)-closed set in \( (X, \tau) \). By assumption, \( f \) (\( F \)) is \( \Lambda^i - \lambda \)-closed in \( (Y, \sigma) \). But \( f(F) = (f^{-1})^{-1}(F) \) and therefore \( f^{-1} \) is \( \Lambda^i \)-continuous on \( Y \).

**Definition 4.16:** (i) A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be a \( \lambda^* \)-closed map if the image of \( f \) (\( A \)) is \( \lambda \)-closed in \( (Y, \sigma) \) for every \( \lambda \)-closed set \( A \) in \( (X, \tau) \).

(ii) A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be a \( \Lambda^i \)-\( \lambda^* \)-closed map if the image \( f \) (\( A \)) is \( \Lambda^i \)-\( \lambda \)-closed in \( (Y, \sigma) \) for every \( \Lambda^i \)-\( \lambda \)-closed set \( A \) in \( (X, \tau) \).
(iii) A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a $\Lambda - gc$–closed if the image $f(A)$
is $\Lambda - g$–closed in $(Y, \sigma)$ for every $\Lambda - g$–closed set $A$ in $(X, \tau)$.

**Example 4.17**

(i) Let $X = Y = \{a, b, c, d, e\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b, c\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a, f(b) = c, f(c) = d, f(d) = b$ and $f(e) = e$. Then $f$ is $\Lambda^* - gc$–closed as well as $\Lambda^{\ast} - gc$–closed.

**Remark 4.18:**

Since every closed set is a $\Lambda^\times - gc$–closed set we have every $\Lambda^{\ast} - gc$–closed map is a $\Lambda^\times - gc$–closed map.

The converse is not true in general as seen from the following example.

**Example 4.19:**

(i) Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b, c\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a, f(b) = b, f(c) = d, f(d) = c$. Then $f$ is $\Lambda^\times - gc$–closed but not $\Lambda^{\ast} - gc$–closed, because for the $\Lambda^\times - gc$–closed set $\{d\}$ in $(X, \tau)$ $f(\{d\}) = c$ which is not a $\Lambda^{\ast} - gc$–closed set in $(Y, \sigma)$.

**Proposition 4.20:** For any bijection $f : (X, \tau) \to (Y, \sigma)$ the following are equivalent:

(i) $f^{-1}: (Y, \sigma) \to (X, \tau)$ is $\Lambda^\times - irresolute$

(ii) $f$ is a $\Lambda^\times c$–open map and

(iii) $f$ is a $\Lambda^\times c$–closed map.

**Proof:** Similar to proposition 4.13.

**Definition 4.21:**

Let $A$ be a subset of $X$. A mapping $r : X \to A$ is called a $\Lambda^\times$–continuous retraction if $r$ is a $\Lambda^\times$–continuous and the restriction if $r$ is a $\Lambda^\times$–continuous and the restriction $r_A$ is the identity mapping on $A$.

**Definition 4.22:** A topological space $(X, \tau)$ is called a $\Lambda^\times$–Hausdorff if for each pair $x, y$ of distinct points of $X$ there exists $\Lambda^\times$–neighborhoods $U_1$ and $U_2$ of $x$ and $y$, respectively, that are disjoint.
Theorem 4.24:
Let $A$ be a subset of $X$ and $r : X \to A$ be a $\Lambda^d$-continuous retraction. If $X$ is $\Lambda^d$-Hausdorff, then $A$ is a $\Lambda^d$-closed set of $X$.

Proof:
Suppose that $A$ is not $\Lambda^d$-closed. Then there exists a point $x$ in $X$ such that $x \in \text{clos}^d(A)$ but $x \not\in A$. It follows that $r(x) \neq x$ because $r$ is $\Lambda^d$-continuous retraction. Since $X$ is $\Lambda^d$-Hausdorff, there exists disjoint $\Lambda^d$-open sets $U$ and $V$ in $X$ such that $x \in U$ and $r(x) \in V$. Now let $W$ be an arbitrary $\Lambda^d$-neighborhood of $x$. Then $W \cap U$ is a $\Lambda^d$-neighborhood of $x$. Since $x \in \text{clos}^d(A)$, we have $(W \cap U) \cap A \neq \emptyset$. Therefore there exists a point $y$ in $W \cap U \cap A$. Since $y \in A$, we have $r(y) = y \in U$ and hence $r(y) \not\in V$. This implies that $r(W) \not\subseteq V$ because $y \in W$. This is contrary to the $\Lambda^d$-continuity of $r$. Consequently, $A$ is a $\Lambda^d$-closed set of $X$.

Theorem 4.25
Let $\{X_i : i \in I\}$ be any family of topological space. If $f : X \to \pi_{\mathcal{I}}$ is a $\Lambda^d$-continuous mapping, then $\pi_{\mathcal{I}}^{-1}(X_i)$ is $\Lambda^d$-continuous for each $i \in I$, where $\pi_{\mathcal{I}}$ is the projection of $\pi_{\mathcal{J}}$ on to $X_i$.

Proof: We shall consider a fixed $i \in I$. Suppose $U_i$ is an arbitrary open set in $X_i$ and by proposition 3.6, we have $A \subseteq \pi_{\mathcal{I}}^{-1}(U_i)$ and hence $\pi_i^{-1}(U_i)$ is open in $\pi_{\mathcal{I}}^{-1}(X_i)$. Since $f$ is $\Lambda^d$-continuous, we have $f^{-1}(\pi_i^{-1}(U_i)) = (\pi_i \circ f)^{-1}(U_i)$ is $\Lambda^d$-open in $\pi_{\mathcal{I}}^{-1}(X_i)$. Therefore $\pi_{\mathcal{I}}^{-1}(U_i)$ is $\Lambda^d$-continuous.

Proposition 4.26:
A mapping $f : (X, \tau) \to (Y, \sigma)$ is $\Lambda^d$-closed if and only if $\text{clos}^d(f(A)) \subseteq f(\text{clos}(A))$ for every subset $A$ of $(X, \tau)$.

Proof: Suppose that $f$ is $\Lambda^d$-closed and $A \subseteq X$. Then $f(\text{clos}(A))$ is $\Lambda^d$-closed in $(Y, \sigma)$. We have $f(A) \subseteq f(\text{clos}(A))$ and by proposition 3.6, $\text{clos}^d(f(A)) \subseteq \text{clos}^d(f(\text{clos}(A))) = f(\text{clos}(A))$. Conversely, let $A$ be any closed set in $(X, \tau)$. By hypothesis and proposition 3.6, we have $A = \text{clos}(A)$ and so $f(A) = f(\text{clos}(A)) \supseteq \text{clos}^d(f(A))$. i.e., $f(A)$ is $\Lambda^d$-closed and hence $f$ is $\Lambda^d$-closed.
Theorem 4.27:
A mapping \( f : (X, \tau) \to (Y, \sigma) \) is \( \lambda \)-closed if and only if \( f \) is both \( \Lambda \)-g-closed and \( \Lambda^\lambda \)-closed.

Proof: Let \( V \) be a closed set in \( (X, \tau) \). As \( f \) is \( \lambda \)-closed, \( f(V) \) is a \( \lambda \)-closed in \( (Y, \sigma) \). By Proposition 3.10, \( f(V) \) is a \( \Lambda \)-g-closed and \( \Lambda^\lambda \)-closed set. Hence \( f \) is \( \Lambda \)-g-closed and \( \Lambda^\lambda \)-closed.

Conversely, let \( V \) be closed in \( (X, \tau) \). As \( f \) is \( \lambda \)-closed and \( \Lambda^\lambda \)-closed \( f(V) \) is both \( \lambda \)-g-closed and \( \Lambda^\lambda \)-closed set. Hence \( f(V) \) is \( \lambda \)-closed by Proposition 3.10.

5. \( \Lambda^\lambda \)-Homeomorphisms.

In this section we introduce and study two new homeomorphisms namely \( \Lambda^\lambda \)-homeomorphism and \( \Lambda^\lambda \)-homeomorphism.

Definition 5.1:
A bijection \( f : (X, \tau) \to (y, \sigma) \) is called \( \lambda \)-homeomorphism if \( f \) is both \( \lambda \)-continuous and \( \lambda \)-open.

Proposition 5.2: Every homeomorphism is a \( \lambda \)-homeomorphism.

Proof: Follows from definitions.

The converse of the Proposition 5.2 need not be true as see from the following example.

Example 5.3:
Let \( X = Y = \{a, b, c, d\}, \tau = \{\phi, X, \{a\}, \{a, b, c\}\} \) and \( \sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}\} \). Define a map \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = a, f(b) = c, f(c) = d \) and \( f(d) = b \). Then \( f \) is \( \lambda \)-homeomorphism but not a homeomorphism, because it is not continuous.

Thus, the class of \( \lambda \)-homeomorphisms properly contains the class of homeomorphism.

Definition 5.4: A bijection \( f : (X, \tau) \to (Y, \sigma) \) is called \( \Lambda^\lambda \)-homeomorphism if \( f \) is both \( \Lambda^\lambda \)-continuous and \( \Lambda^\lambda \)-open.

Proposition 5.5: Every \( \lambda \)-homeomorphism is a \( \Lambda^\lambda \)-homeomorphism

proof: Follows from definitions.

The converse of the Proposition 5.5 need not be true as seen from the following example.

Example 5.6:
Let \( X = Y = \{a, b, c, d\}, \tau = \{\phi, X, \{a\}, \{a, b, c\}\} \) and \( \sigma = \{\phi, Y, \{a\}, \{a, b\}, \{a, c, d\}\} \). Define a map \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = c, f(b) = d, f(c) = a \) and \( f(d) = b \). Then \( f \) is \( \Lambda^\lambda \)-homeomorphism but not a homeomorphism. Because it is not a \( \lambda \)-continuous function.
Thus, the class of $\Lambda^{\delta}$-homeomorphisms properly contains the class of $\lambda$-homeomorphisms.

**Definition 5.7:**
A bijection $f : (X, \tau) \to (Y, \sigma)$ is called $\Lambda^{\delta} - \lambda$-homeomorphism if $f$ is both $\Lambda^{\delta} - \lambda$-continuous and $\Lambda^{\delta} - \lambda$-open.

**Proposition 5.8:**
Every $\Lambda^{\delta} - \lambda$-homeomorphism is a $\Lambda^{\delta}$-homeomorphism but not conversely.

**Proof:** Follows from definitions.

The converse of the Proposition 5.8 need not be true as seen from the following example.

**Example 5.9:**
The function $f$ in 5.3 is $\Lambda^{\delta}$-homeomorphism but not $\Lambda^{\delta} - \lambda$-homeomorphism. Because $f$ is not $\Lambda^{\delta} - \lambda$-continuous.

Thus, the class of $\Lambda^{\delta} - \lambda$-homeomorphisms property contains the class of $\Lambda^{\delta} - \lambda$-homeomorphisms.

**Proposition 5.10:** Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection $\Lambda^{\delta}$-continuous map. Then the following statements are equivalent:

(i) $f$ is a $\Lambda^{\delta}$-open map

(ii) $f$ is a $\Lambda^{\delta}$-homeomorphism

(iii) $f$ is a $\Lambda^{\delta}$-closed map.

**Proof:** (i) $\iff$ (ii) Follows from the definition.

(i) $\iff$ (iii) Follows from proposition 3.13.

**Proposition 5.11:** Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection $\Lambda^{\delta} - \lambda$-continuous map. Then the following statements are equivalent:

(i) $f$ is a $\Lambda^{\delta} - \lambda$-open map

(ii) $f$ is a $\Lambda^{\delta} - \lambda$-homeomorphism

(iii) $f$ is a $\Lambda^{\delta} - \lambda$-closed map.

**Proof:** (i) $\iff$ (ii) Follows from the definition.

(i) $\iff$ (iii) Follows from proposition 3.15.

The composition of two $\Lambda^{\delta}$-homeomorphism maps need not be a $\Lambda^{\delta}$-homeomorphism as can be seen from the following example.
Example 5.12:

Let

\[ X = Y = \{ a, b, c, d, e \}, \tau = \{ \phi, X, \{ a, b \}, \{ b, c \}, \{ a, b, c \} \}, \]
\[ \sigma = \{ \phi, Y, \{ c, d \}, \{ b, c, d \}, \{ a, c, d \} \} \]

and \( \eta = \{ \phi, Z, \{ a \}, \{ b, a \}, \{ b, c \}, \{ a, b, c \} \} \), Define a map \( f : (X, \tau) \to (Y, \sigma) \) by

\[ f(a) = e, f(b) = b, f(c) = a, f(d) = c, f(e) = d \quad \text{and} \quad g : (Y, \sigma) \to (Z, \eta) \]

by

\[ g(a) = a, g(b) = b, g(c) = e, g(d) = d, g(e) = c. \]

Then \( f \) and \( g \) are \( \Lambda^2 \)-homeomorphisms but their composition \( gof : (X, \tau) \to (Z, \eta) \) is not a \( \Lambda^2 \)-homeomorphism, because 

\( (gof)^{-1}(\{a, b\}) = \{b, c\} \) which is not a \( \Lambda^2 \)-open set in \( (z, \eta) \). Therefore \( gof \) is not a \( \Lambda^2 \)-continuous map and so \( gof \) is not a \( \Lambda^2 \)-homeomorphism.

We next introduce a new class of maps called \( \Lambda^2 \)-homeomorphisms which forms a sub class of \( \Lambda^2 \)-homeomorphisms. This class of maps is closed under composition of maps.

Definition 5.13:

A bijection \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( \lambda^* \)-homeomorphism if both

\( f \) and \( f^{-1} \) are \( \lambda \)-irresolute.

We denote the family of all \( \lambda \)-homeomorphisms (resp. \( \lambda^* \)-homeomorphism and homeomorphism) of a topological space \( (X, \tau) \) on to itself by \( \lambda - h(X, \tau) \). (resp. \( \lambda^* - h(X, \tau) \) and \( h(X, \tau) \).

Proposition 5.14:

Every \( \lambda^* \)-homeomorphism is a \( \lambda \)-homeomorphism but not conversely (i.e) for any space \( (X, \tau) \),  

\( \lambda^* - h(X, \tau) \subset \lambda - h(X, \tau) \).

Proof:

It follows from the fact that every \( \lambda \)-irresolute map is a \( \lambda \)-continuous map and the fact that \( \lambda^* \)-open map is \( \lambda \)-open map.

The function \( f \) in example 5.3 is a \( \lambda \)-homeomorphism but not a \( \lambda^* \)-homeomorphism, since for the \( \lambda \)-closed set \( \{a, b\} \) in \( (Y, \sigma) \), \( f^{-1}(\{a, b\}) = \{a, d\} \) which is not \( \lambda \)-closed in \( (X, \tau) \). Therefore \( f \) is not \( \lambda \)-irresolute and so \( f \) is not a \( \lambda^* \)-homeomorphism.

Definition 5.15:

A bijection \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( \Lambda^2^* \)-homeomorphism if both \( f \) and \( f^{-1} \) are \( \Lambda^2 \)-irresolute.
We denote the family of all \( \Lambda^1 \)-homeomorphisms (resp. \( \Lambda^{\ast} \)-homeomorphism) of a topological space \((X, \tau)\) on to itself by \( \lambda - h(X, \tau) \). (resp. \( \Lambda^1 - h(X, \tau) \).

**Proposition 5.16:**

Every \( \Lambda^{\ast} \)-homeomorphism is a \( \Lambda^1 \)-homeomorphism but not conversely (i.e) for any space \((X, \tau)\), 
\( \Lambda^* - h(X, \tau) \subset \Lambda^1 - h(X, \tau) \).

**Proof:**

Follows from theorem 3.5 and the fact that every \( \Lambda^{\ast} \)-open map is \( \Lambda^1 \)-open. The function \( f \) in Example 5.4 is a \( \Lambda^1 \)-homeomorphism but not a \( \Lambda^{\ast} \)-homeomorphism, since for the \( \Lambda^1 \)-closed set \( \{a, b, c\}\) in \((y, \sigma)\), \( f^{-1}(\{a, b, c\}) = \{a, b, c\} \) which is not \( \Lambda^1 \)-closed set in \((X, \tau)\). Therefore \( f \) is not \( \Lambda^1 \)-irresolute and so \( f \) is not a \( \Lambda^{\ast} \)-homeomorphism.

**Theorem 5.17:**

If \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) are \( \Lambda^{\ast} \)-homeomorphism, then their composition \( \text{gof} : (X, \tau) \rightarrow (Z, \eta) \) is also \( \Lambda^{\ast} \)-homeomorphisms.

**Proof:**

Let \( B \) be a \( \Lambda^1 \)-open set in \((Z, \eta)\). Now, \( (\text{gof})^{-1}(B) = f^{-1}(g^{-1}(B)) = f^{-1}(C) \), where \( C = g^{-1}(B) \).

By hypothesis, \( C \) is \( \Lambda^1 \)-open in \((Y, \sigma)\) and so again by hypothesis, \( f^{-1}(C) \) is \( \Lambda^1 \)-open in \((X, \tau)\).

Therefore \( \text{gof} \) is \( \Lambda^1 \)-irresolute. Also for a \( \Lambda^1 \)-open set \( G \) in \((X, \tau)\). We have \( (\text{gof})(G) = g(f(G)) = g(V) \), where \( V = f(G) \). By hypothesis \( f(G) \) is \( \Lambda^1 \)-open in \((Y, \sigma)\) and so again by hypothesis, \( g(f(G)) \) is \( \Lambda^1 \)-open in \((Z, \eta)\) i.e., \( (\text{gof})(G) \) is \( \Lambda^1 \)-open in \((Z, \eta)\) and therefore \( (\text{gof})^{-1} \) is \( \Lambda^1 \)-irresolute. Hence \( \text{gof} \) is a \( \Lambda^{\ast} \)-homeomorphism.

**Theorem 5.18:**

The set \( \Lambda^{\ast} - h(X, \tau) \) is a group under the composition of maps.

**Proof:**

Define a binary operation \( * : \Lambda^{\ast} - h(X, \tau) \times \Lambda^{\ast} - h(X, \tau) \rightarrow \Lambda^{\ast} - h(X, \tau) \) by \( f * g = \text{gof} \) for all \( f, g \in \Lambda^* - h(X, \tau) \) and \( \circ \) is the usual operation of composition of maps. Then by theorem 5.17, \( \text{gof} \in \Lambda^{\ast} - h(X, \tau) \). We know that the composition of maps is associative and the identity map \( I : (X, \tau) \rightarrow (X, \tau) \) belonging to \( \Lambda^1 - h(X, \tau) \) serves as the identity element.
$f \in \Lambda^\ast - h(X, \tau)$, then $f^{-1}\Lambda^\ast - h(X, \tau)$ such that $fof^{-1} = f^{-1}of = I$ and so inverse $(\Lambda^\ast - h(X, \tau), o)$ is a group under the operation of composition of maps.

**Theorem 5.19:**

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\Lambda^\ast$-homeomorphism. Then $f$ induces an isomorphism from the group $\Lambda^\ast - h(X, \tau)$ on to the group $\Lambda^\ast - h(Y, \sigma)$.

**Proof:**

Using the map $g$, we define a map $\varphi_g : \Lambda^\ast - h(X, \tau) \rightarrow \Lambda^\ast - h(Y, \sigma)$ by $\varphi_g(h) = goh h^{-1}$ for every $h \in \Lambda^\ast - h(X, \tau)$. Then $\varphi_g$ is a bijection. Further, for all $h, h_1 \in \Lambda^\ast - h(X, \tau)$, $\varphi_g(h_1oh_2) = go(h_1oh_2) g^{-1} = (goh_1 g^{-1}) o(goh_2 g^{-1}) = \varphi_g(h_1) o \varphi_g(h_2)$.

Therefore, $\varphi_g$ is a homeomorphism and so it is an isomorphism induced by $g$.

**Theorem 5.20:**

$\Lambda^\ast$-homeomorphisms is an equivalence relation in the collection of all topological spaces.

**Proof:** Reflexivity and symmetry are immediate and transitivity follows from Theorem 5.17.

**References**


