Journal of Mathematical Sciences & Computer Applications 1 (2): 72–77, 2011.
doi: 10.5147/jmsca.2011.00

On π-Quasi Irresolute Functions

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Abstract
In this paper we introduce a new class of functions called π-quasi irresolute functions. The notion of π-quasi graphs are introduced and the relationship between π-quasi irresolute functions and π-quasi closed graphs is analysed.

2000 Mathematics Subject Classification : 54C08, 54C10.

Received: February 7, 2011 / Accepted: March 23, 2011

1 Introduction

In 1970, Levine [1] intiated the study of g-closed sets.Over the years this notion has been studied extensively by many topologists. Zaitsev [15] introduced the concept of π-closed sets and defined a class of topological spaces called quasi normal spaces. J.Dontchev and T.Noiri[5] introduced the class of πg-closed sets and obtained a new characterization of quasi-normal spaces. Recently, new classes of functions called regular set-connected [4] have been introduced and investigated. Ekici [6]]extended the concept of regular set-connected functions to almost clopen functions.Saeid Jafari and Noiri [12] introduced α-quasi-irresolute functions and studied the relationships between α-quasi-irresolute functions and graphs.

In this paper we introduce a new class of functions called π-quasi irresolute functions and its fundamental properties are explored.We introduce π-quasi-closed graphs and study the relationships between π-quasi irresolute functions and π-quasi-closed graphs.

2 Preliminaries

Throughout this paper (X,τ) and (Y,σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A in X respectively.

A subset A of a space X is called regular open [11] if A = int(cl (A)).The family of all regular open (resp regular closed, clopen , semi-open) sets of X is denoted by RO(X) (resp RC(X), CO(X), SO(X)). A finite union of regular open sets is called a π-open set. .The family of all π-open sets of X is denoted by πO(X). The π-interior of A is the union of all π-open sets of X contained in A and it is denoted by π-int(A).The complement of a π-open set is called π-closed

A subset A is said to be semi-open[10] if A$\subset$ cl(int(A)).A point x is said to be θ-semi cluster point if a subset A of X is such that cl(U)$\cap$ A$\neq\phi$ for every U $\in$ SO(X,x). The set of all θ-semi cluster points of A is called a θ-semi closure of a set and it is denoted by θ-s-cl(A). A subset A is called θ-semi closed [8] if A = θ-scl(A). The complement of a θ-semi closed set is called θ-semi open.The union of all α-open sets contained in S is called the α-interior of S and is denoted by α-int(S). We set α(X,x) = {U/ x $\in$ U $\in$ α(X) }

We recall the following definitions, which are useful in the sequel.

Definition 2.1. A Space X is said to be

1. πT_1 [7] if for each pair of distinct points x and y of X , there exist π-open sets U and V containing x and y respectively such that y$\notin$ U and x $\notin$ V.

2. π-Lindelof [7] if every cover of X by π-open sets has a countable subcover.

3. S-closed [14] if every cover of X by semi-open sets of X admits a finite subfamily, whose closures cover X.

4. countably S-closed [3] if every countable cover of X by regular closed sets has a finite subcover.

5. S-Lindelof [2] if every cover of X by regular closed sets has a countable subcover.

Definition. 2.2. A function f : X $\to$ Y is said to be

1. θ- irresolute [9] if for each x $\in$X and each V $\in$SO (Y,f(x)), there exist U $\in$ SO (X,x) such that f (cl(U)) $\subset$ cl(V).

2. regular set-connected [4] if f^{-1}(V) is clopen in X for every regular open set V of Y.

3. π-set connected [7] if f^{-1}(V) $\in$ CO(X) for every V $\in$ πO(Y).

4. (θ-s)-continuous [8] if for each x $\in$ X and each V $\in$ SO(Y, f(x)), there exists an open set U in X containing x such that f(U) $\subset$ Cl(V).

5. α-quasi-irresolute [12](briefly α-q-i) if for each x $\in$ X and each V $\in$ SO(Y, f(x)), there exists U $\in$ α(X, x) such that f(U) $\subset$ CI(V).

Remark 2.2. It should be noted that the following implications hold : [12]

regular set-connected $\Rightarrow(\theta, s)$-continuous $\Rightarrow\alpha$-quasi irresolute.

3 π-Quasi-irresolute functions

Definition 3.1. A function f : X $\to$Y is called π-Quasi -irresolute if for each x $\in$ X and each V $\in$ SO(Y, f(x)) there exist a π-open set U in X containing x such that f(U) $\subset$ cl(V) .

Remark 3.2. The following implications can be easily established

π-Quasi irresolute $\Rightarrow$ (θ, s)-continuous.

Theorem 3.3. *Suppose that πO(X) is closed under arbitrary union , then the following are equivalent for a function f : X $\to$ Y*

1. f is π- quasi -irresolute.

2. f^{-1}(V) $\subset$ πint (f^{-1}(cl(V))) for every V $\in$SO(Y).

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3. The inverse image of a regular closed set of Y is π-open.

4. The inverse image of a regular open set of Y is π-closed.

5. The inverse image of a θ-semi open set of Y is π-open.

6. f^{-1}(int(cl (G))) is π-closed. for every open subset G of Y.

7. f^{-1}(cl (int(F))) is π-open for every closed subset F of Y.

Proof. (1) $\Rightarrow$(2) :Let V $\in$SO (Y) and x $\in f^{-1}(V)$. Then f(x) $\in$ V . Since f is π-quasi irresolute, there exist a π-open set U in X containing x such that f (U) $\subset$ cl(V). It follows that x $\in$ U $\subset f^{-1}$(cl (V)).Hence x $\in\pi$ int(f^{-1}(cl(V)) . Therefore $f^{-1}(V) \subset \pi$-int(f^{-1}(cl(V)).

(2) $\Rightarrow$(3) : Let F be any regular closed set of Y. Since F $\in$ SO (Y) , then by (2) , f^{-1}(F) $\subset \pi$ int(f^{-1}(F)). This shows that f^{-1}(F) is π-open.

(3) $\Leftrightarrow$(4) : is obvious.

(4) $\Rightarrow$(5) : This follows from our assumption and the fact that any θ-semi open set is a union of regular closed sets.

(5) $\Rightarrow$(1). Let x $\in$ X and V $\in$ SO(Y) . Since cl (V) is θ-semi open in Y by (5) there exist a π-open set U in X containing x such that x $\in$ U $\subset$(f^{-1}(cl(V)). hence f (U) $\subset$ cl (V). Hence f is π-quasi irresolute

(4) $\Rightarrow$ (6) let G be a open subset of Y. Since int(cl (G)) is regular open , then by (4), f^{-1}(int (cl (G))) is π- closed

(6) $\Rightarrow$(4) is obvious.

(3) $\Rightarrow$(7) is similar as (4) $\Leftrightarrow$(6) . ☐

Definition 3.4. A Space X is said to be

1. π-compact if every cover of X by π-open sets has a finite subcover.

2. countably π-compact if every countable cover of X by π-open sets has a finite subcover.

3. π-Hausdroff (πT_2) if for each pair of distinct points x and y in X , there exist U $\in \pi$O(X,x) and V$\in\pi$O(Y,y) such that U $\cap$ V = ϕ.

Remark 3.5. Here it should be noted that following implications hold:
π-Hausdroff space $\Rightarrow$ Hausdorff space .

Lemma 3.6. *Let S be an open subset of a space (X, τ) then If U is π-open in X , then so is U $\cap$ S in the subspace (S , τ_s)*

Theorem 3.7. *If f : X $\rightarrow$ Y is a π-quasi-irresolute function and A is any open subset of X , then the restriction f / A : A $\rightarrow$Y is π-quasi irresolute function.*

Proof. Let F $\in$ RC (Y). Then by theorem 3.3 , $f^{-1}(F) \in \pi O$(X).Since A is any open set in X , (f / A $)^{-1}$(F) = f^{-1}(F) $\cap$ A $\in \pi O$(A) . Therefore f / A is π-quasi irresolute function. ☐

Definition 3.8. A space X is said to be

1. S-Urysohn [1] if for each pair of distinct points x and y in X , there exist U $\in$ SO(X,x) and V $\in$ SO(X,Y) such that cl (U) $\cap$ cl (V) = ϕ .

2. Weakly Hausdorff [13] if each element of X is an intersection of regular closed sets.

Theorem 3.9. *If f:X$\rightarrow$Y is π-quasi irresolute injection and Y is S-Urysohn, then X is π-Hausdorff.*

Proof. Suppose that Y is S-Urysohn. By the injectivity of f, it follows that f(x) $\neq$ f(y) for any distinct points x and y in X. Since Y is S-Urysohn, there exist V $\in$ SO (Y, f(x)) and W $\in$ SO (Y, f(y)) such that cl (V) $\cap$ cl (W) = ϕ . Since f is a π-quasi irresolute function, there exist π-open sets U and G in X containing x and y respectively, such that f (U) $\subset$ cl (V) and f (G) $\subset$ cl (W)and we have U $\cap$ G = ϕ . Hence X is π-Hausdorff. ☐

Theorem 3.10. *If* f:X$\rightarrow$Y *is π-quasi irresolute injection and Y is weakly Hausdorff then X is* πT_1*.*

Proof. Suppose that Y is weakly Hausdorff. For any distinct points x and y there exist V , W $\in$RC (Y) such that f(x) $\in$ V and f(y) $\notin$V , f(x) $\notin$ W and , f(y)$\in$W . Since f is π-quasi irresolute injection, by theorem 3.2 , f^{-1}(V) and f^{-1}(W) are π-open subsets of X such that x $\in$ f^{-1}(V), x $\notin$ f^{-1}(W) and y $\in$ f^{-1}(W). This shows that X is πT_1. ☐

Theorem 3.11. *If f , g : X $\rightarrow$ Y are π-quasi irresolute functions and Y is S-Urysohn, then E = { x $\in$ X / f(x) = g(x) } is closed in X.*

Proof. If x $\in$ X - E . Then f(x) $\neq$ g(x).Since Y is S-Urysohn, there exist V $\in$ SO (Y, f(x)) and W $\in$ SO (Y,g(x)) such that cl(V) $\cap$ cl(W) = ϕ. Since f and g are π-quasi irresolute , there exist π-open sets U and G , which are open sets in X such that f(U) $\subset$ cl(V) and g(G) $\subset$ cl(W) . Set O = U $\cap$ G . Then O is open in X.

f(O) $\cap$ g(O) = f (U$\cap$ G) $\cap$ g (U $\cap$G) $\subset$ f(U) $\cap$ g(G) $\subset$ cl(V) $\cap$ cl (W) = ϕ. O is an open set and O $\cap$ E = ϕ . Therefore x $\notin$ cl (E) . E is closed in X. ☐

Theorem 3.12. *Let f : X $\rightarrow$Y be a function and g : X $\rightarrow$X$\times$ Y the graph function of f defined by g(x) = (x, f(x)) for every x $\in$ X. If g is π-quasi irresolute , then f is π-quasi irresolute .*

Proof. Let F $\in$ RC(Y), then X$\times$ F = X $\times$ cl (int(F)) = cl (int(X) $\times$cl(int(F)) = cl (int (X$\times$F)). Therefore X$\times$F $\in$RC (X$\times$Y). It follows from theorem 3.3, that f^{-1}(F) = g^{-1}(X$\times$ F) is π-open in X.Thus f is π-quasi-irresolute. ☐

Definition 3.13. A function f : X $\rightarrow$Y is said to be

1. π-open if the image of each π-open set is π-open.
2. π-irresolute if for each x $\in$ X and π-open set V in Y ,containing f(x), there exist a π-open set U in X ,containing x, such that f(U) $\subset$V.

Theorem 3.14. *Let f : X $\rightarrow$Y and g : X $\rightarrow$ Y be functions. Then the following hold :*

1. If f is π-irresolute and g is π-quasi irresolute then g $\circ$ f : X $\rightarrow$ Z is π-quasi irresolute.
2. If f is π-quasi irresolute and g is θ-irresolute, then g $\circ$ f : X $\rightarrow$ Z is π-quasi irresolute.

Proof. 1) Let x $\in$ V and W be a semi-open set in Z containing (g$\circ$f)(x). since g is π-quasi irresolute, there exist a π-open set V in Y containing f(x) such that g(V) $\subset$cl(W). Since f is π- irresolute, there exist π-open set U in X such that f(U) $\subset$ V.

This shows that (g$\circ$f)(U) $\subset$ cl (W). Therefore g$\circ$ f is π-quasi irresolute.

2) Let x $\in$ X and W be a semi-open set in Z containing g$\circ$f (x). Since g is θ-irresolute, there exist V $\in$ SO (Y, f(x)) such that g (cl(V)) $\subset$ cl(W). Since f is π-quasi irresolute there exist a π-open set U(X,x) such that f(U) $\subset$ cl(V). Therefore, we have (g$\circ$f)(U) $\subset$ cl (W) . This shows that (g$\circ$f) is π-quasi irresolute. ☐

Theorem 3.15. *If f: X $\rightarrow$Y is a π-open surjective function and g : Y $\rightarrow$ Z is a function such that g $\circ$ f : X $\rightarrow$Z is π-quasi irresolute , then g is π-quasi irresolute.*

Proof. Suppose that x and y are in X and Y respectively such that f(x) = y. Let W be a semi-open set in Z containing g$\circ$f (x) . Then there exist U $\in\pi O(X,x)$ such that g(f(U)) $\subset$cl(W). Since f is π-open , then f(U) $\in$ πO(Y,y) such that g(f(U)) $\subset$ cl(W). This implies that g is π-quasi irresolute. ☐

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4 π-quasi-closed graph

Definition 4.1. The graph G(f) of a function f : X $\rightarrow$Y is said to be π-quasi-closed if for each (x,y) $\in$ X$\times$Y - G(f), there exist U $\in \pi O(X,x)$ and V $\in$ SO(Y,y) such that (U$\times$cl(V)) $\cap$ G(f) = ϕ.

Lemma 4.2. *The following properties are equivalent for a graph G(f) of a function f : X $\rightarrow$ Y .*

1. The graph G (f) is π-quasi-closed in X$\times$Y .
2. For each point (x,y) $\in$ X$\times$ Y - G(f) , there exist U $\in \pi O$(X,x) and V $\in$ SO(Y,y) such that f(U) $\cap$ cl(V) = ϕ.
3. For each point (x,y) $\in$ X $\times$ Y - G(f) , there exist U $\in \pi O$(X,x) and F $\in$ RC(Y,y) such that f(U) $\cap$ F = ϕ.

Proof. (1) $\Rightarrow$ (2) follows from the definition and the fact that for any subset A $\subset$X, B $\subset$Y (A$\times$B) $\cap$ G(f) = ϕ iff f(A) $\cap$ B = ϕ.

(2) $\Rightarrow$(3)follows from the fact that cl(V) $\in$RC(Y) for any V $\in$SO (Y).

(3)$\Rightarrow$(1). It is obvious since every regular closed set is semi-open and closed. □

Theorem 4.3. *If f: X $\rightarrow$ Y is π-quasi-irresolute and Y is S-Urysohn, then G(f) is π-quasi-closed in X$\times$ Y.*

Proof. Let (x,y) $\in$X$\times$Y - G(f). It follows that f(x) $\neq$ y. Since Y is S-Urysohn, there exist V $\in$ SO(Y,f(x)) and W $\in$ SO(Y,y) such that cl(V) $\cap$ cl(W) = ϕ. Since f is π-quasi-irresolute, there exist π-open set U(X,x) such that f(U) $\subset$ cl(V). Therefore , f(U) $\cap$ cl(W) $\subset$ cl(V) $\cap$ cl(W) = ϕ. and G(f) is π-quasi-closed in X $\times$Y. □

Theorem 4.4. *If f : X $\rightarrow$Y is surjective and G(f) is π-quasi-closed then Y is weakly Hausdorff.*

Proof. Let $y_1$and $y_2$be any distinct points of Y. Since f is surjective f(x) = $y_1$for some x $\in$X and (x, y_2) $\in$X$\times$Y - G(f). By lemma 4.2, there exist U $\in \pi O$(X,x) and F $\in$ RC (Y,y_2) such that f(U)$\cap$ F = ϕ. Hence $y_1 \notin$ F . This implies Y is weakly hausdorff. □

Theorem 4.5. *If f : X $\rightarrow$Y is π-quasi irresolute with a π-quasi-closed graph then X is π- Hausdorff.*

Proof. Let x , y be any two distinct points of X. Since f is injective we have f(x) $\neq$f(y) and thus (x, f(y)) $\in$ X$\times$Y - G(f) . Since G(f) is π-quasi-closed , there exist U $\in \pi O$(X,x) and V $\in$ SO(Y,f(y)) such that f(U) $\cap$ cl(V) = ϕ. Since f is π-quasi irresolute there exist G $\in \pi O$(X,y) such that f(G) $\subset$ cl(V). Therefore , we have f(U) $\cap$ f(G) = ϕ and hence U $\cap$ G = ϕ. This shows that X is π-T_2. □

Theorem 4.6. *If f : X $\rightarrow$Y is a π-quasi irresolute, closed function from a normal space X onto a space Y, then any two disjoint θ-semi closed subsets of Y can be separated.*

Proof. Let $F_1$and F_2 be any distinct θ-semi closed sets of Y. Since f is π-quasi-irresolute, f^{-1}(F_1) and f^{-1}(F_2) are disjoint π-closed sets of X and hence closed. By normality of X, there exist open sets U_1, U_2in X such that f^{-1}(F_1) $\subset$ U_1 and f^{-1}(F_2) $\subset$ $U_2$and $U_1 \cap U_2 = \phi$. Let V_i= Y - f (X-U_i) for i = 1,2.

Since f is closed , the sets $V_1$and V_2 are open in Y and $F_i \subset V_i$ for i = 1,2 . Since $U_1$and U_2 are disjoint and f^{-1}(F_i) $\subset$ U_i for i = 1, 2, we obtain $V_1 \cap V_2 = \phi$. This shows that $F_1$and F_2 are seperated. □

Definition 4.7. A topological space (X, τ) is said to be π-connected if X cannot be written as the disjoint union of two non empty π-open sets.

Theorem 4.8. *If f : X$\rightarrow$ Y is π-quasi irresolute surjection and X is π-connected , then Y is connected.*

Proof. Suppose that Y is not connected space . There exist non-empty disjoint open sets $V_1$and V_2 such that Y = $V_1 \cup V_2$. Therefore $V_1$and V_2 are clopen in Y. Since f is π-quasi irresolute $f^{-1}(V_1)$ and f^{-1} (V_2) are π-open in X. Moreover f^{-1}(V_1) and f^{-1}(V_2) are non-empty disjoint and X = f^{-1}(V_1) $\cup$ f^{-1}(V_2). This contradicts that Y is not connected. □

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