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# Edge Product Number of Graphs in Paths

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## Abstract

**A graph G (V, E) is said to be a sum graph if there exists a bijective labeling from the vertex set V to a set S of positive integers such that $(x \times y) \in E$ if and only if $f(x) + f(y) \in S$. In this paper, for a given graph G (V, E), the edge function, the edge product function and the edge product graph are introduced and studied. The edge product number of a graph is defined and the edge product numbers of paths is found.**

**Keywords:** Edge function, edge product function, edge product graph, edge product number of graph and optimal edge product function.

## 1. Introduction

Harary F introduced the notation of sum graph [6,7]. He defined sum number of a graph as a minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph. He also conjectured that every tree T with $\zeta(T) = 0$ is a caterpillar in [6]. Chen Z conjectured that all trees are $\int \sum$ - graphs [2,3]. For more on sum graphs and exclusive sum number can be found in [1,5]. Ellingnham proved that the sum number of a tree is one [4]. The sum number of a complete graph $K_n$ with $n \geq 4$ vertices gives as S(K) = (2n - 3) in [9]. The sum number of paths is found in [6]. For a detailed account on variations of sum graphs one can refer to Gallian [8]. We want to introduce the edge as well as the product analogue of sum graph. This paper gives an idea about edge product graphs and the edge analogue of product graphs. We also characterize the edge product number of connected graphs. A graph is said to be an edge product graph if the edges of G can be labeled with distinct positive integers such that the product of all the labels of the edges incident on a vertex is again an edge label of G and if the product of any collection of edges is a label of an edge in G, then they are incident on a vertex. In this paper, for a given graph G, the edge product number of graphs is defined and investigate the edge product numbers of paths.

## 2. Edge Product Graph

*Definition 2.1:* Let G be a given graph. A bijection f: E $\to$ P where P is a set of positive integers is called an edge function of G. Define $F(v) = \prod\{f(e)\text{: e is incident on v}\}$ on V. Then the function F is called the edge product function of the edge function f. The graph G is said to be an edge product graph if there exists an edge function f: E $\to$ P such that the function f and its corresponding edge product function F on V satisfies that $F(v) \in P$ for every $v \in V$ and if $e_1, e_2, \ldots, e_p \in E$ such that $f(e_1)\times f(e_2)\times\ldots\times f(e_p) \in P$, then the edges $e_1, e_2, \ldots, e_p$ are incident on a vertex.

*Example 2.2:* Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ be the vertex set and $E = \{v_1v_2, v_2v_3, v_3v_4, v_2v_5, v_3v_6, v_7v_8\}$ be the edge set of G. The edge function f: E $\to$ P is defined by $f(v_1v_2) = 2^6$, $f(v_2v_3) = 2^3$, $f(v_3v_4) = 2^4$, $f(v_2v_5) = 2^5$, $f(v_3v_6) = 2^7$ and $f(v_7v_8) = 2^{14}$. The corresponding edge product function F is given by F is given by $F(v_1) = 2^6$, $F(v_2) = 2^{14}$, $F(v_3) = 2^{14}$, $F(v_4) = 2^4$, $F(v_5) = 2^5$, $F(v_6) = 2^7$, $F(v_7) = 2^{14}$ and $F(v_8) = 2^{14}$. Clearly G is an edge product graph.

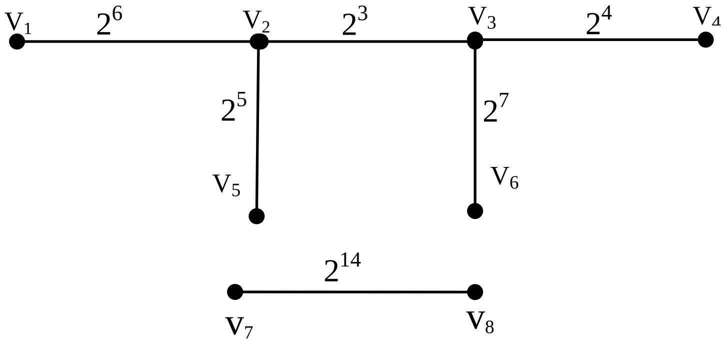


Figure 1

*Note 2.3:* If G is an edge product graph then $K_2$ is a component of G.

## 3. Edge Product Number of a Graph

Edge product number of a graph is a minimum number r of $K_2$ components that must be added to G so that the resulting graph is the edge product graph. Thus the graph $G \cup rK_2$ is an edge product graph for minimum r then the number r is called the edge product number of G and is denoted by EPN(G). For any connected graph G other than $K_2$, EPN(G) $\geq$ 1. Let EPN(G) = r. An edge function f: E $\rightarrow$ P and its corresponding edge product function F which make $G \cup rK_2$ an edge product graph are respectively called an optimal edge function and an optimal edge product function of G. Let $E = E_1 \cup E_2$ where $E_1$ is the edge set of G and $E_2$ that of $rK_2$. Then, EPN(G) = Cardinality of the set $\{F(v): v \in V, F(v) \notin f(E_1)\}$. If $F(V) \cap f(E_1) = \Phi$ then F is said to be outer edge product function and if $F(V) \cap f(E_1) \neq \Phi$, then F is said to be an inner edge product function. The range of F has atleast r elements. It has exactly r elements if and only if F is outer edge product function.

## 4. Edge Product Number of Paths

A walk is called a trail if all the edges appearing in the walk are distinct. It is called a path if all the vertices are distinct. We present here the edge product number of paths. Let $P_q$ be a path on q vertices with $V = \{v_1, v_2, \ldots, v_q, v_{(q+1)}\}$ and $E = \{v_i v_{(i+1)}: 1 \leq i \leq q\}$ be the vertex set and edge set respectively. The following figure shows that $EPN(P_2) = EPN(P_3) = EPN(P_4) = EPN(P_6) = 1$.

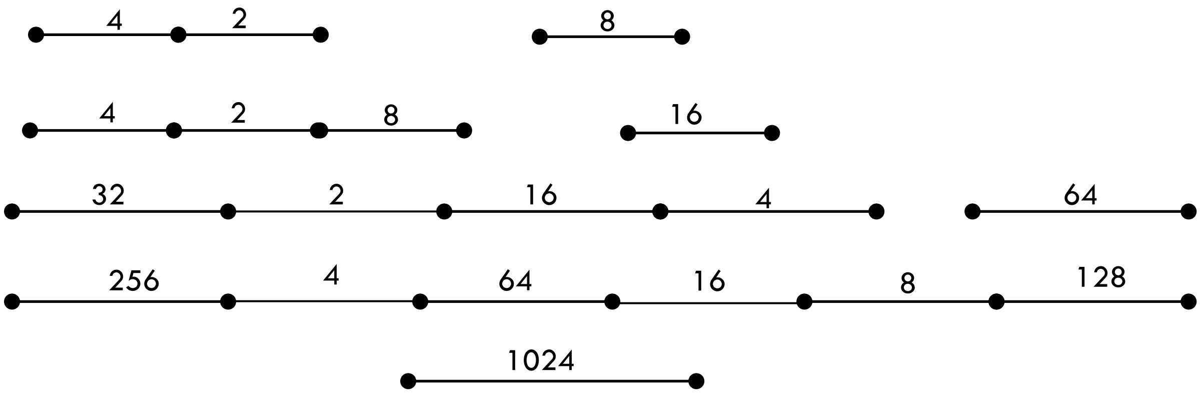


Figure 2

**Theorem 4.1:** $EPN(P_q) = 2$ for some $q = 5$.

**Proof:** Assume that $EPN(P_q) = 1$ for some $q = 5$ then $(P_5 \cup K_2)$ is an edge product graph. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6, w_1, w_2\}$ and $E = \{v_i v_{(i+1)}: 1 \le i \le 5\} \cup \{w_1 w_2\}$ be the vertex set and edge set of G respectively. The elements of the set $P = \{a_1, a_2, a_3, a_4, a_5, b\}$. The mapping $f: E \to P$ is an optimal edge function and F is the optimal edge product function of f.

Let the optimal edge function f is defined by $f(v_i v_{(i+1)}) = a_i$ for $1 \le i \le 5$ and $f(w_1 w_2) = b$. Then the optimal edge product function F is defined by

$F(v_1) = f(v_1 v_2) = a_1$

$F(v_i) = f(v_{(i-1)} v_i) \times f(v_i v_{(i+1)}) = a_{(i-1)} \times a_i$ for $2 \le i \le 5$

$F(v_6) = f(v_5 v_6) = a_5$ and $F(w_1) = F(w_2) = b$.

Since, $a_{(i-1)} \times a_i \ne a_i \times a_{(i+1)}$ for $2 \le i \le 5$ and $F(v_2) \ne F(v_3)$, $F(v_3) \ne F(v_4)$, $F(v_4) \ne F(v_5)$. The vertices $v_1$ and $v_6$ are pendent vertices. Then $F(v_3)$ can be $f(v_1 v_2)$ and $F(v_4)$ can be $f(v_5 v_6)$. Since the function F is into P, we get $F(v_2) = F(v_5) = F(w_1) = F(w_2) = b$, $F(v_3) = f(v_1 v_2) = a_1$ and $F(v_4) = f(v_5 v_6) = a_5$. Therefore $(a_1 \times a_2) = (a_4 \times a_5)$, $(a_2 \times a_3) = a_1$ and $(a_3 \times a_4) = a_5$. That is $(a_2 \times a_3) \times a_2 = a_4 \times (a_3 \times a_4) \Rightarrow a_2 = a_4$. This is a contradiction to our assumption that the elements of P are distinct. Thus $EPN(P_q) \ge 2$ for some $q = 5$. The graph $(P_5 \cup K_2)$ is an edge product graph and $EPN(P_5) = 2$ shown below.

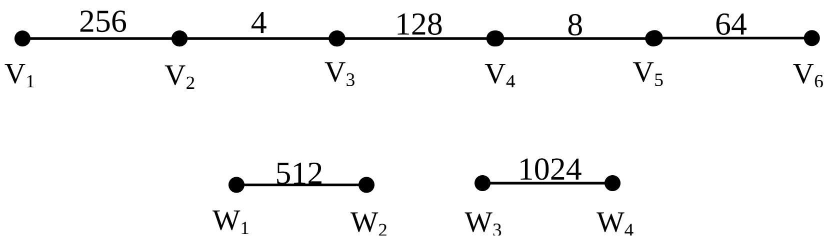


Figure 3

**Theorem 4.2:** $EPN(P_q) = 2$ for some $q \geq 7$.

**Proof:** Consider $P_q$ is a path on some $q \geq 7$ and $EPN(P_q) = 1$. The graph $(P_q \cup K_2)$ is an edge product graph with $V = \{v_1, v_2, \ldots, v_q, v_{(q+1)}, w_1, w_2\}$ and $E = \{v_i v_{(i+1)}: 1 \leq i \leq q\} \cup \{w_1 w_2\}$. Let $f: E \to P$ be an optimal edge function and F be its corresponding optimal edge product function of G. The edge product graph G has no triangles. But G has four pendent vertices namely $v_1, v_{(q+1)}, w_1, w_2$ and the three pendent edges namely $v_1 v_2$, $v_q v_{(q+1)}$ and $w_1 w_2$. Then we have $F(V) \subseteq \{f(v_1 v_2), f(v_q v_{q+1}), f(w_1 w_2)\}$ and also $F(v_1) = f(v_1 v_2)$, $F(v_3)$ can be $f(v_1 v_2)$. Similarly $F(v_{(q+1)}) = f(v_q v_{(q+1)})$, $F(v_{(q-1)})$ can be $f(v_q v_{(q+1)})$. Therefore $F(v) = f(w_1 w_2)$ for all other vertices of v. But $F(v_3) = F(v_4) = f(w_1 w_2)$ for some $q \geq 7$.

Hence $F(v_3) = f(v_2 v_3) \times f(v_3 v_4) \neq f(v_3 v_4) \times f(v_4 v_5) = F(v_4)$ which is a contradiction. Thus we obtain the result that $EPN(P_q) \geq 2$ for some $q \geq 7$. Suppose $(P_q \cup 2K_2)$ for some $q \geq 7$ with $V = \{v_1, v_2, \ldots, v_q, v_{(q+1)}, w_1, w_2, w_3, w_4\}$ and $E = \{v_i v_{(i+1)}: 1 \leq i \leq q\} \cup \{w_1 w_2, w_3 w_4\}$ then there may arise two cases.

Case (1) when q is odd

Take $q = (2p + 1)$ for some $p \geq 3$. Consider $A = p^2 + 1 + [p(p+1)/2]$ and

$P = \{2^{p+i}: 1 \leq i \leq p\} \cup \{2^{A+k}: 0 \leq k \leq p\} \cup \{(2^{A+2p}), (2^{A+2p+1})\}$.

Define the edge function $f: E \to P$ by

$f(v_{2i} v_{(2i+1)}) = 2^{p+i}$ for $1 \leq i \leq p$

$f(v_{(2i+1)} v_{(2i+2)}) = 2^{A+p-i}$ for $1 \leq i \leq p$

$f(w_1 w_2) = 2^{A+2p}$ and $f(w_3 w_4) = 2^{A+2p+1}$

The corresponding edge product function F is defined by

$F(v_1) = f(v_1 v_2) = 2^{A+p}$

$F(v_{2i}) = f(v_{(2i-1)} v_{2i}) \times f(v_{2i} v_{(2i+1)})$ for $1 \leq i \leq p$

$= 2^{A+p-i+1} \times 2^{p+i} = 2^{A+2p+1} = f(w_3 w_4)$

$F(v_{2i+1}) = f(v_{2i} v_{(2i+1)}) \times f(v_{(2i+1)} v_{(2i+2)})$ for $1 \leq i \leq p$

$= 2^{p+i} \times 2^{A+p-i} = 2^{A+2p} = f(w_1 w_2)$

$F(v_{2p+2}) = f(v_{(2p+1)} v_{(2p+2)}) = 2^A$

$F(w_1) = F(w_2) = f(w_1 w_2) = 2^{A+2p}$ and $F(w_3) = F(w_4) = f(w_3 w_4) = 2^{A+2p+1}$

Therefore the four elements of F are the elements of P, namely $2^A$, $2^{A+p}$, $2^{A+2p}$ and $2^{A+2p+1}$. Hence the function F is into P. Also $2^{p+1}, 2^{p+2}, \ldots, 2^{2p}, 2^A, 2^{A+1}, \ldots, 2^{A+p}, 2^{A+2p}$ and $2^{A+2p+1}$ are the elements of P in ascending order. Then the elements of P have the following three conditions:

(i) $2^{p+1} \times 2^{p+2} = 2^{2p+3} > 2^{2p}$

(ii) $2^{p+1} \times 2^{p+2} \times \ldots \times 2^{2p} = 2^{(p^2 + p(p+1))/2} < 2^{A}$

(iii) $2^{p+1} \times 2^{A} > 2^{A+p}$

If $f(e_1) \times f(e_2) \times \ldots \times f(e_r) = P$ where the r edges $e_1, e_2, \ldots, e_r \in E$ and $r > 1$, then P is either $2^{A+2p}$ or $2^{A+2p+1}$.Now the elements of P are divided into three sets, namely $P_1 = \{2^{p+1}, 2^{p+2}, \ldots, 2^{p+p} = 2^{2p}\}$, $P_2 = \{2^{A}, 2^{A+1}, 2^{A+2}, \ldots, 2^{A+p}\}$and $P_3 = \{2^{A+2p}, 2^{A+2p+1}\}$. Therefore $P = P_1 \cup P_2 \cup P_3$ and the elements have the following four properties:

(i) Product of all the elements of $P_1 < 2^{A} < 2^{A+2p} < 2^{A+2p+1}$

(ii) $2^{p+1} \times 2^{p+2} \times 2^{A} > 2^{A+2p+1}$

(iii) $2^{A} \times 2^{A+1} > 2^{A+2p+1}$

(iv) $2^{p+1} \times 2^{A+2p} > 2^{A+2p+1}$

If the product of a collection of more than one element of P is either$2^{A+2p}$ or $2^{A+2p+1}$ then the collection contains exactly one element from $P_1$ and one element from $P_2$. Then for $1 \leq i \leq p$, the elements $2^{(A+2p)-(p+i)}$ and $2^{(A+2p+1)-(p+i)}$ are uniquely determined. Thus the 2p collections gives the products $F(v_i)$ for $2 \leq i \leq (2p+1)$. If $f(e_1) \times f(e_2) \times \ldots \times f(e_r) \in P$ then $r = 2$ and the edges $e_1$ and $e_2$ are incident on a vertex. Therefore, for odd integers $q \geq 7$, $(P_q \cup 2K_2)$ is an edge product graph. Thus $EPN(P_q) \leq 2$ for some $q \geq 7$. This proves that $EPN(P_{(2p+1)}) = 2$ for some $p \geq 3$. The following figure shows that the graph $(P_9 \cup 2K_2)$ is an edge product graph and $EPN(P_9) = 2$.

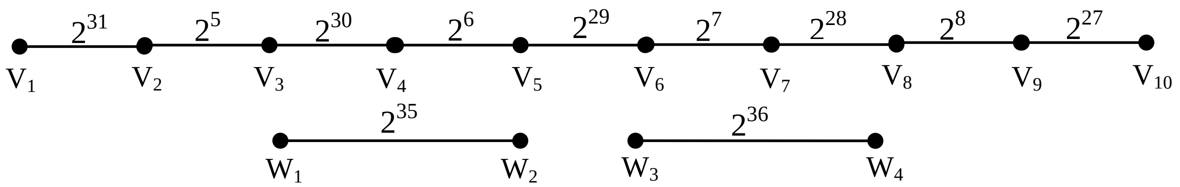


Figure 4

Case (2) when q is even

If $q = 2p$ for some $p \geq 4$. Consider $B = \{p^2 + [p(p-1)/2]\}$ and

$P = \{2^{p-1+i}: 1 \leq i \leq p\} \cup \{2^{B+i}: 1 \leq i \leq p\} \cup \{2^{B+2p-1}, 2^{B+2p}\}$.

Define the edge function $f: E \rightarrow P$ by

$f(v_{2i}v_{(2i+1)}) = 2^{p-1+i}$ for $1 \leq i \leq p$

$f(v_{(2i-1)}v_{2i}) = 2^{B+p+1-i}$ for $1 \leq i \leq p$

$f(w_1w_2) = 2^{B+2p-1}$ and $f(w_3w_4) = 2^{B+2p}$

The corresponding edge product function F of f is defined by

$F(v_1) = f(v_1v_2) = 2^{B + p}$

$F(v_{2i}) = f(v_{(2i - 1)}v_{2i}) \times f(v_{2i}v_{(2i + 1)})$ for $1 \leq i \leq p$

$= 2^{B + p + 1 - i} \times 2^{p - 1 + i} = 2^{B + 2p} = f(w_3w_4)$

$F(v_{(2i + 1)}) = f(v_{2i}v_{(2i + 1)}) \times f(v_{(2i + 1)}v_{(2i + 2)})$ for $1 \leq i \leq (p - 1)$

$= 2^{p - 1 + i} \times 2^{B + p - i} = 2^{B + 2p - 1} = f(w_1w_2)$

$F(v_{(2p + 1)}) = 2^{2p - 1}$; $F(w_1) = F(w_2) = f(w_1w_2) = 2^{B + 2p - 1}$ and $F(w_3) = F(w_4) = f(w_3w_4) = 2^{B + 2p}$. Therefore, F has only four elements which are the elements of P, namely $2^{B+p}$, $2^{2p - 1}$, $2^{B + 2p - 1}$ and $2^{B + 2p}$. Hence F is into P. For $q = 2p$, $(P_q \cup 2K_2)$ is an edge product graph for every $p \geq 4$. Thus $EPN(P_{2p}) = 2$ for some $p \geq 4$. The following figure shows that $(P_8 \cup 2K_2)$ is an edge product graph and $EPN(P_8) = 2$.

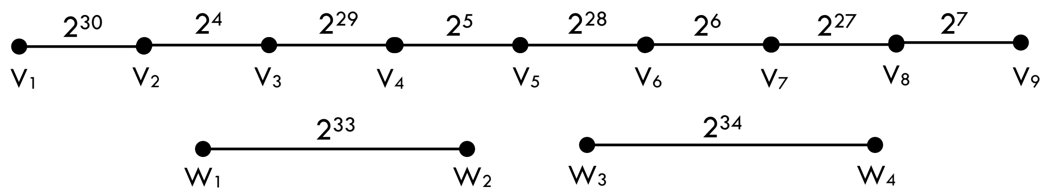


Figure 5

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